Nonlinear Signal Processing (2004/2005)

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Riemannian Metrics

Definition [Tensor] Let V be a n-dimensional vector space over $\mathbb{R}$. A tensor Φ of order r on V is a multilinear (i.e., linear in each argument) map

$$\Phi \,:\, \underbrace{V \times \cdots \times V}_{r} \to \mathbb{R}.$$

Thus,

$$\Phi(\cdots, av + bw, \cdots) = a\,\Phi(\cdots, v, \cdots) + b\,\Phi(\cdots, w, \cdots)$$

for all $a, b \in \mathbb{R}$ and $v, w \in V$.

The set of tensors of order r on V is denoted by $T^r(V)$.

A tensor of order $r = 1$ is usually called a covector, and the set of covectors is usually denoted by V^* and termed the dual space of V (note: $V^* = T^1(V)$).

Example 1 [A tensor of order 1 (covector) on $\mathsf{M}(n, \mathbb{R})$] The map

$$\Phi \,:\, \mathsf{M}(n, \mathbb{R}) \to \mathbb{R} \qquad \Phi(X) = \mathsf{tr}(X)$$

is a covector on $\mathsf{M}(n, \mathbb{R})$.

Example 2 [A tensor of order 2 on $\mathbb{R}^n$] Let $A \in \mathsf{M}(n, \mathbb{R})$. The map

$$\Phi \,:\, \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \qquad \Phi(x, y) = x^T A y$$

is a tensor of order 2 on $\mathbb{R}^n$.

The map

$$\Psi \,:\, \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \qquad \Psi(x, y) = x^T A y + 1$$

is <u>not</u> a tensor.

Example 3 [A tensor of order 3 on $\mathsf{M}(n, \mathbb{R})$] The map

$$\Phi \,:\, \mathsf{M}(n, \mathbb{R}) \times \mathsf{M}(n, \mathbb{R}) \times \mathsf{M}(n, \mathbb{R}) \to \mathbb{R} \qquad \Phi(A, B, C) = \mathsf{tr}\,(ABC)$$

is a tensor of order 3 on $\mathsf{M}(n, \mathbb{R})$.

Example 4 **[A tensor of order n on $\mathbb{R}^n$]** The map

$$\Phi : \underbrace{\mathbb{R}^n \times \cdots \times \mathbb{R}^n}_{n} \to \mathbb{R} \qquad \Phi(x_1, x_2, \ldots, x_n) = \det\left([\, x_1\, x_2\, \cdots\, x_n \,]\right)$$

is a tensor of order n on $\mathbb{R}^n$.

Theorem **[The vector space of tensors of order r]** Let V be a n-dimensional vector space over $\mathbb{R}$. Let r be a positive integer. With the operations of addition and multiplication by real numbers defined as

$$\begin{aligned} (\Phi_1 + \Phi_2)(v_1, \ldots, v_r) &= \Phi_1(v_1, \ldots, v_r) + \Phi_2(v_1, \ldots, v_r) \\ (\alpha\Phi)(v_1, \ldots, v_r) &= \alpha\, \Phi(v_1, \ldots, v_r) \end{aligned}$$

for $\Phi, \Phi_1, \Phi_2 \in T^r(V)$, $\alpha \in \mathbb{R}$, and $v_1, \ldots, v_r \in V$, the set $T^r(V)$ is a vector space of dimension n^r.

Lemma **[Pullback of tensors]** Let V, W be vector spaces over $\mathbb{R}$ and let $F_* : V \to W$ be a linear map. Then, for each positive integer r, the map $F^* : T^r(W) \to T^r(V)$, defined as

$$\left(F^*\Phi\right)(v_1, \ldots, v_r) = \Phi\left(F_*(v_1), \ldots, F_*(v_r)\right)$$

for all $\Phi \in T^r(W)$, is a linear map between the vector spaces $T^r(W)$ and $T^r(V)$.

Example 1 **[Pullback of a tensor]** Consider the linear map

$$F_* : \mathbb{R}^n \to \mathsf{M}(n, \mathbb{R}) \qquad F_*(x) = \mathsf{Diag}(x) = \begin{bmatrix} x_1 & & & \\ & x_2 & & \\ & & \ddots & \\ & & & x_n \end{bmatrix}$$

and the tensor of order 2 on $\mathsf{M}(n, \mathbb{R})$ given by

$$\Phi : \mathsf{M}(n, \mathbb{R}) \times \mathsf{M}(n, \mathbb{R}) \to \mathbb{R} \qquad \Phi(A, B) = \mathsf{tr}(AB).$$

The pullback of Φ by F_*, denoted $F^*\Phi$, is the tensor of order 2 on $\mathbb{R}^n$ given by

$$\begin{aligned}
(F^*\Phi)(x,y) &= \Phi(F_*(x)F_*(y)) \\
&= \mathsf{tr}\left(\mathsf{Diag}(x)\,\mathsf{Diag}(y)\right) \\
&= \mathsf{tr}\begin{bmatrix} x_1y_1 & & & \\ & x_2y_2 & & \\ & & \ddots & \\ & & & x_ny_n \end{bmatrix} \\
&= x^Ty.
\end{aligned}$$

Example 2 [Another pullback of a tensor] Consider the linear map

$$F_* : \mathsf{M}(p,q,\mathbb{R}) \to \mathsf{M}(n,\mathbb{R}) \qquad F_*(X) = DXE,$$

where $D \in \mathsf{M}(n,p,\mathbb{R})$ and $E \in \mathsf{M}(q,n,\mathbb{R})$ are fixed matrices, and the tensor of order 3 on $\mathsf{M}(n,\mathbb{R})$ given by

$$\Phi : \mathsf{M}(n,\mathbb{R})\times\mathsf{M}(n,\mathbb{R})\times\mathsf{M}(n,\mathbb{R}) \to \mathbb{R} \qquad \Phi(A,B,C) = \mathsf{tr}\left(A\otimes B\otimes C\right).$$

The pullback of Φ by F_*, denoted $F^*\Phi$, is the tensor of order 3 on $\mathsf{M}(p,q,\mathbb{R})$ given by

$$\begin{aligned}
(F^*\Phi)(X_1,X_2,X_3) &= \Phi(F_*(X_1),F_*(X_2),F_*(X_3)) \\
&= \mathsf{tr}\left(DX_1E\otimes DX_2E\otimes DX_3E\right).
\end{aligned}$$

Definition [Product of tensors] Let V be a vector space over $\mathbb{R}$. Let $\Phi \in T^r(V)$ and $\Psi \in T^s(V)$. The product of Φ and Ψ, denoted $\Phi\otimes\Psi$ is a tensor of order $r+s$ defined by

$$(\Phi\otimes\Psi)\,(v_1,\ldots,v_r,v_{r+1},\ldots,v_{r+s}) = \Phi(v_1,\ldots,v_r)\Psi(v_{r+1},\ldots,v_{r+s}).$$

Example 1 [Product of tensors] Consider the tensors

$$\Phi : \underbrace{\mathbb{R}^n\times\cdots\times\mathbb{R}^n}_{n} \to \mathbb{R} \qquad \Phi(x_1,x_2,\ldots,x_n) = \det\left[\,x_1\,x_2\,\cdots\,x_n\,\right]$$

and

$$\Psi : \mathbb{R}^n \times \mathbb{R}^n \to \mathbb{R} \qquad \Psi(x,y) = x^T M y,$$

where $M \in \mathsf{M}(n,\mathbb{R})$ is a fixed matrix.

Note that Φ is a tensor of order n and Ψ is a tensor of order 2. Thus, $\Phi \otimes \Psi$ is a tensor of order $n+2$ on $\mathbb{R}^n$ and it is given by

$$\begin{aligned}(\Phi \otimes \Psi)(x_1, \dots, x_n, x_{n+1}, x_{n+2}) &= \Phi(x_1, \dots, x_n)\, \Psi(x_{n+1}, x_{n+2}) \\ &= \det\left[\, x_1 \ \cdots \ x_n \,\right] x_{n+1}^T M x_{n+2}.\end{aligned}$$

Theorem [Product of tensors] Let V be a n-dimensional vector space over $\mathbb{R}$. The product of tensors

$$T^r(V) \times T^s(V) \to T^{r+s}(V) \qquad (\Phi, \Psi) \mapsto \Phi \otimes \Psi,$$

is bilinear and associative:

$$\begin{aligned}(a\Phi_1 + b\Phi_2) \otimes \Psi &= a\,(\Phi_1 \otimes \Psi) + b\,(\Phi_2 \otimes \Psi) \\ \Phi \otimes (a\Psi_1 + b\Psi_2) &= a\,(\Phi \otimes \Psi_1) + b\,(\Phi \otimes \Psi_2) \\ (\Phi \otimes \Psi) \otimes \Omega &= \Phi \otimes (\Psi \otimes \Omega).\end{aligned}$$

If $\omega^1, \omega^2, \dots, \omega^n$ is a basis of $V^* = T^1(V)$, then

$$\left\{\omega^{i_1} \otimes \omega^{i_2} \otimes \cdots \otimes \omega^{i_r} \;:\; 1 \le i_1, i_2, \dots, i_r \le n\right\}$$

is a basis of $T^r(V)$. That is, each $\Phi \in T^r(V)$ can be written as

$$\Phi = \sum_{(i_1,\dots,i_r) \in I^r} a_{i_1 \cdots i_r}\, \omega^{i_1} \otimes \omega^{i_2} \otimes \cdots \otimes \omega^{i_r},$$

where $I = \{1, 2, \dots, n\}$ and $a_{i_1 \cdots i_r} \in \mathbb{R}$ are constants (uniquely determined by Φ).

Furthermore, if $F_* : V \to W$ is a linear map of vector spaces, then $F^*(\Phi \otimes \Psi) = (F^*\Phi) \otimes (F^*\Psi)$.

Example 1 **[Expansion of a tensor]** Let $V = \mathbb{R}^3$ and the canonical basis $\{e_1, e_2, e_3\} = \{(1,0,0), (0,1,0), (0,0,1)\}$. The dual space V^* is spanned by the dual basis $\{\omega^1, \omega^2, \omega^3\}$ where

$$\omega^i : V \to \mathbb{R} \qquad \omega^i(e_j) = \delta^i_j = \begin{cases} 0 & , \quad i \neq j \\ 1 & , \quad i = 1 \end{cases}.$$

In equivalent terms,

$$\omega^i : V \to \mathbb{R} \qquad \omega^i(x) = e_i^T x.$$

For example, $\omega^1 \otimes \omega^3$ is the tensor of order 2 given by

$$\begin{aligned} (\omega^1 \otimes \omega^3)(x, y) &= \omega^1(x)\, \omega^3(y) \\ &= (x^T e_1)\,(e_3^T y) \\ &= x^T (e_1 e_3^T) y \\ &= x^T \begin{bmatrix} 0 & 0 & 1 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} y. \end{aligned}$$

Consider the tensor of order 2 on $\mathbb{R}^3$ given by

$$\Phi : \mathbb{R}^3 \times \mathbb{R}^3 \to \mathbb{R} \qquad \Phi(x, y) = x^T \begin{bmatrix} 1 & -1 & 2 \\ 4 & 2 & 0 \\ 3 & 2 & 2 \end{bmatrix} y.$$

Then, we can expand Φ as

$$\begin{aligned} \Phi = \; & \omega^1 \otimes \omega^1 - \omega^1 \otimes \omega^2 + 2\,\omega^1 \otimes \omega^3 + \\ & 4\,\omega^2 \otimes \omega^1 + 2\,\omega^2 \otimes \omega^2 + \\ & 3\,\omega^3 \otimes \omega^1 + 2\,\omega^3 \otimes \omega^2 + 2\,\omega^3 \otimes \omega^3. \end{aligned}$$

Equivalently, if Φ is any given tensor of order 2 on $\mathbb{R}^3$ we can find the coefficients Φ_{ij} in the expansion

$$\begin{aligned} \Phi = \; & \Phi_{11}\, \omega^1 \otimes \omega^1 + \Phi_{12}\, \omega^1 \otimes \omega^2 + \Phi_{13}\, \omega^1 \otimes \omega^3 + \\ & \Phi_{21}\, \omega^2 \otimes \omega^1 + \Phi_{22}\, \omega^2 \otimes \omega^2 + \Phi_{23}\, \omega^2 \otimes \omega^3 + \\ & \Phi_{31}\, \omega^3 \otimes \omega^1 + \Phi_{32}\, \omega^3 \otimes \omega^2 + \Phi_{33}\, \omega^3 \otimes \omega^3 \end{aligned}$$

by calculating $\Phi_{ij} = \Phi(e_i, e_j)$.

Example 2 **[Tensors of order 2 as matrices and vice-versa]** Let V be any real n-dimensional vector space. Once we fix a basis $\{E_1, E_2, \dots, E_n\}$ for V, any tensor Φ of order 2 can be represented by a $n \times n$ matrix M_Φ:

$$\Phi\left(x^i E_i, y^j E_j\right) = x^i y^j \Phi(E_i, E_j) = x^T \underbrace{\begin{bmatrix} \Phi(E_1, E_1) & \cdots & \Phi(E_1, E_n) \\ \vdots & \ddots & \vdots \\ \Phi(E_n, E_1) & \cdots & \Phi(E_n, E_n) \end{bmatrix}}_{M_\Phi} y.$$

This equivalence (w.r.t to a fixed basis $\{E_1, \dots, E_n\}$) is denoted by

$$\Phi \sim M_\Phi.$$

Definition **[Smooth tensor fields]** A smooth tensor field of order r on a smooth manifold M is a map Φ which assigns to each $p \in M$ an element $\Phi_p \in T^r(T_pM)$ such that: for any smooth vector fields $X_1, \dots, X_r$ defined on M, the function

$$\Phi(X_1, \dots, X_r) : M \to \mathbb{R} \qquad p \mapsto \Phi_p\left(X_{1p}, \dots, X_{rp}\right)$$

is smooth.

The set of smooth tensor fields of order r on M is denoted by $\mathcal{T}^r(M)$ and it is a vector space under pointwise addition of tensors and multiplication by elements of $\mathbb{R}$.

▷ *Intuition: a smooth tensor field on M is a smooth assignment of a tensor Φ_p to each tangent space T_pM*

Example 1 **[Canonical smooth covector fields on $\mathbb{R}^n$]** The n covector fields $dx^1, \dots, dx^n$ defined, at each $p \in \mathbb{R}^n$, to be the dual basis of the canonical basis

$$\frac{\partial}{\partial x^1}\bigg|_p, \frac{\partial}{\partial x^2}\bigg|_p, \dots, \frac{\partial}{\partial x^n}\bigg|_p$$

of $T_p\mathbb{R}^n$, are smooth covector fields on $\mathbb{R}^n$.

Example 2 [**A smooth covector field on** $\mathbb{R}^3$] The covector field

$$\omega = (x^2 - y + z)\, dx + (\sin(xy) - 3)\, dy + xe^y\, dz$$

is a smooth covector field on $\mathbb{R}^3$.

At $p = (1, 0, 1)$, we have the covector $\omega_{(1,0,1)} : T_{(1,0,1)}\mathbb{R}^3 \to \mathbb{R}$ given by

$$\omega_{(1,0,1)} \left(a \frac{\partial}{\partial x}\bigg|_{(1,0,1)} + b \frac{\partial}{\partial y}\bigg|_{(1,0,1)} + c \frac{\partial}{\partial z}\bigg|_{(1,0,1)} \right) = 2a - 3b + c.$$

Example 3 [**A smooth function induces a smooth covector field**] Let M be a smooth manifold and $f \in C^\infty(M)$. The covector field df defined, at each $p \in M$, by

$$df|_p(X_p) = X_p f, \qquad X_p \in T_pM,$$

is smooth. It is called the differential of f.

For the special case $M = \mathbb{R}^n$, we have

$$df|_p = \frac{\partial f}{\partial x^1}(p)\, dx^1|_p + \frac{\partial f}{\partial x^2}(p)\, dx^2|_p + \cdots + \frac{\partial f}{\partial x^n}(p)\, dx^n|_p.$$

For instance, if

$$f : \mathbb{R}^3 \to \mathbb{R} \qquad f(x, y, z) = \cos(xz) + y^2,$$

then

$$df = -z \sin(xz) dx + 2y dy - x \sin(xz) dz.$$

Definition [**Product of tensor fields**] Let M be a smooth manifold. Let Φ and Ψ be tensor fields on M order r and s, respectively. The product of Φ and Ψ, denoted $\Phi \otimes \Psi$ is the tensor field of order $r + s$ on M defined for each $p \in M$ as $(\Phi \otimes \Psi)_p = \Phi_p \otimes \Psi_p$.

Theorem [**Product of smooth tensor fields**] Let M be a n-dimensional smooth manifold. The product of tensor fields just defined takes smooth tensor fields to smooth tensor fields. Moreover, the map

$$\mathcal{T}^r(M) \times \mathcal{T}^s(M) \to \mathcal{T}^{r+s}(M) \qquad (\Phi, \Psi) \mapsto \Phi \otimes \Psi$$

is bilinear and associative.

If $\omega^1, \omega^2, \ldots, \omega^n$ is a basis of $\mathcal{T}^*(M) = \mathcal{T}^1(M)$ (that is, $\omega^1_p, \ldots, \omega^n_p$ is a basis of $T^*_p M$ at each $p \in M$), then each $\Phi \in \mathcal{T}^r(M)$ can be written as

$$\Phi = \sum_{(i_1,\ldots,i_r)\in I^r} f_{i_1 \cdots i_r}\, \omega^{i_1} \otimes \omega^{i_2} \otimes \cdots \otimes \omega^{i_r},$$

where $I = \{1, 2, \ldots, n\}$ and $f_{i_1 \cdots i_r} : M \to \mathbb{R}$ are smooth functions (uniquely determined by Φ).

Lemma [Pullback of smooth tensor fields] Let M, N be smooth manifolds and $F : M \to N$ a smooth map. Then, each smooth tensor field $\Phi \in \mathcal{T}^r(N)$ determines a smooth tensor field $F^*\Phi \in \mathcal{T}^r(M)$ by the formula

$$(F^*\Phi)_p (X_{1p}, \ldots, X_{rp}) = \Phi_{F(p)} (F_*(X_{1p}), \ldots, F_*(X_{rp}))\,.$$

The map $F^* : \mathcal{T}^r(N) \to \mathcal{T}^r(M)$ thus defined is linear. Moreover,

$$F^* (\Phi \otimes \Psi) = (F^*\Phi) \otimes (F^*\Psi)$$

for $\Phi \in \mathcal{T}^r(M)$ and $\Psi \in \mathcal{T}^s(M)$.

Example 1 [A simple pullback] Let $M = \mathbb{R}^m$, $N = \mathbb{R}^n$ and consider a smooth map

$$F : \mathbb{R}^m \to \mathbb{R}^n \qquad F(x) = (F^1(x), F^2(x), \ldots, F^n(x)).$$

Let $1 \leq j \leq n$. The pullback of dy^j by F is given by

$$F^* \, dy^j = a^j_i dx^j$$

and each smooth function $a^j_i : \mathbb{R}^n \to \mathbb{R}$ can be found as

$$\begin{aligned} a^j_i &= (F^* dy^j)(\partial_i) \\ &= dy^j(F_*\partial_i) \\ &= dy^j \left(\partial_i F^k \partial_k\right) \\ &= \partial_i F^k (dy^j)(\partial_k) \\ &= \partial_i F^j. \end{aligned}$$

Thus,

$$F^* dy^j = \partial_i F^j \, dx^i.$$

For instance, if

$$F \,:\, \mathbb{R}^2 \to \mathbb{R}^3 \qquad F(u,v) = (u^2 + v, \sin(uv), ue^v)$$

with coordinates (u, v) on $\mathbb{R}^2$ and (x, y, z) on $\mathbb{R}^3$, then

$$\begin{aligned} F^*dx &= 2u\,du + dv \\ F^*dy &= v\cos(uv)\,du + u\cos(uv)\,dv \\ F^*dz &= e^v\,du + ue^v\,dv. \end{aligned}$$

Example 2 [**A pullback of a tensor of order** 2] Let $M = \mathbb{R}^m$, $N = \mathbb{R}^n$ and consider a smooth map

$$F \,:\, \mathbb{R}^m \to \mathbb{R}^n \qquad F(x) = (F^1(x), F^2(x), \ldots, F^n(x)).$$

Let $\Phi = \Phi_{ij}\, dy^i \otimes dy^j$ be a smooth tensor field of order 2 on $\mathbb{R}^n$. Note that each

$$\Phi_{ij} \,:\, \mathbb{R}^n \to \mathbb{R}$$

is a smooth function.

The pullback of Φ by F is the tensor field of order 2 on $\mathbb{R}^m$ given by

$$\begin{aligned} F^*\Phi &= F^*\left(\Phi_{ij}\,dy^i \otimes dy^j\right) \\ &= F^*\Phi_{ij}\,F^*dy^i \otimes F^*dy^j \\ &= (\Phi_{ij}\circ F)\,\left(\partial_k F^i dx^k\right) \otimes \left(\partial_l F^j dx^l\right) \\ &= (\Phi_{ij}\circ F)\,\partial_k F^i \partial_l F^j dx^k \otimes dx^l. \end{aligned}$$

In a more compact notation: if $\Phi_y \sim M(y)$ then

$$(F^*\Phi)_p \sim DF(p)^T M(F(p)) DF(p).$$

Definition [**Riemannian manifold**] A Riemannian manifold is a smooth manifold M on which is defined a Riemannian metric g, that is, a smooth tensor of order 2 which is symmetric ($g(X_p, Y_p) = g(Y_p, X_p)$ for all $X_p \in T_pM$) and positive-definite ($g(X_p, X_p) = 0$ if and only if $X_p = 0$).

A smooth manifold M with Riemannian metric g is usually denoted by (M, g).

Example 1 [**The canonical Riemannian metric on** $\mathbb{R}^n$] The canonical Riemannian metric on $\mathbb{R}^n$ is given by

$$g(\partial_i|_p, \partial_j|_p) = \delta_i^j.$$

Thus, for each $p \in \mathbb{R}^n$, we have

$$g_p \sim \begin{bmatrix} 1 & & & \\ & 1 & & \\ & & \ddots & \\ & & & 1 \end{bmatrix}.$$

Example 2 [**Another Riemannian structure on** $\mathbb{R}^n$] Let Σ be a fixed $n \times n$ positive definite matrix. Define a tensor field g of order 2 on $\mathbb{R}^n$ as $g(\partial_i|_p, \partial_j|_p) = \Sigma_{ij}$ or, equivalently,

$$g_p \sim \Sigma \qquad \text{for all } p \in \mathbb{R}^n.$$

The tensor field g is a valid Riemannian metric on $\mathbb{R}^n$.

Example 3 [**A Riemannian structure on** $\mathbb{R}^3$] Define a tensor field g of order 2 on $\mathbb{R}^3$ as

$$g_{(x,y,z)} \sim \begin{bmatrix} e^{2x+yz} & & \\ & 2 - \cos(z) & \\ & & y^2 + 1 \end{bmatrix}.$$

The tensor field g is a valid Riemannian metric on $\mathbb{R}^3$.

Example 4 [**A Riemannian structure on** $\mathsf{P}(n, \mathbb{R})$] Since $\mathsf{P}(n, \mathbb{R})$ is an open subset of $\mathsf{S}(n, \mathbb{R})$ we have the identification

$$T_P\mathsf{P}(n, \mathbb{R}) \simeq \mathsf{S}(n, \mathbb{R}) \qquad \text{for all } P \in \mathsf{P}(n, \mathbb{R}).$$

Consider the inner-product at $T_P\mathsf{P}(n,\mathbb{R})$:

$$g_P\,(\Delta,\Omega) = \mathsf{tr}(\Omega^T\Delta) \qquad \text{for} \qquad \Delta,\Omega \in T_P\mathsf{P}(n,\mathbb{R}) \simeq \mathsf{S}(n,\mathbb{R}).$$

The tensor field g is a valid Riemannian metric on $\mathsf{P}(n,\mathbb{R})$.

Example 5 [Left-invariant Riemannian metrics on Lie groups] Let G be a lie group. Let g_e be any inner-product on T_eG. We can propagate this inner-product to all the group G through left-translations, that is, to each $a \in G$, we assign the inner-product g_a on T_aG as

$$g_a(X_a, Y_a) = g_e(L_{a^{-1}*}X_a, L_{a^{-1}*}Y_a),$$

where

$$L_b\,:\,G \to G \qquad L_b(x) = b\cdot x$$

denotes left translation.

The resulting Riemannian metric $g = \langle\cdot,\cdot\rangle$ is left-invariant:

$$\langle X_a, Y_a\rangle = \langle L_{b*}(X_a), L_{b*}(Y_a)\rangle \qquad \text{for all } a, b \in G.$$

For instance, consider the Lie group $\mathsf{GL}(n,\mathbb{R})$. Since $\mathsf{GL}(n,\mathbb{R})$ is an open subset of $\mathsf{M}(n,\mathbb{R})$ we have the identification

$$T_A\mathsf{GL}(n,\mathbb{R}) \simeq \mathsf{M}(n,\mathbb{R}) \qquad \text{for all } A \in \mathsf{GL}(n,\mathbb{R}).$$

Consider the usual inner-product at $T_{I_n}\mathsf{GL}(n,\mathbb{R})$:

$$\langle\Delta,\Omega\rangle = \mathsf{tr}(\Omega^T\Delta) \qquad \text{for} \qquad \Delta,\Omega \in T_{I_n}\mathsf{GL}(n,\mathbb{R}) \simeq \mathsf{M}(n,\mathbb{R}).$$

By left-translation the inner-product at $T_A\mathsf{GL}(n,\mathbb{R})$ is given by:

$$\langle\Delta,\Omega\rangle = \mathsf{tr}(\Omega^T\left(AA^T\right)^{-1}\Delta) \qquad \text{for} \qquad \Delta,\Omega \in T_A\mathsf{GL}(n,\mathbb{R}) \simeq \mathsf{M}(n,\mathbb{R}).$$

Lemma [Riemannian submanifolds] Let (M, g_M) be a Riemannian manifold. Let $N \subset M$ be a submanifold with inclusion map $\iota\,:\,N \to M$. Then, $g_N = \iota^* g_M$ is a Riemannian metric on N.

Example 1 [The canonical Riemannian metric on $\mathsf{S}^2(\mathbb{R})$] Let

$$F : W \subset \mathbb{R}^2 \to \mathsf{S}^2(\mathbb{R}) \qquad F(\theta,\varphi) = (\cos\theta\,\sin\varphi,\, \sin\theta\,\sin\varphi,\, \cos\varphi)$$

be a local parameterization of the unit-sphere $\mathsf{S}^2(\mathbb{R})$ where

$$W = \{(\theta,\varphi) \,:\, 0 < \theta < 2\pi, 0 < \varphi < \pi/2\}\,.$$

In these coordinates, the canonical Riemannian metric on $\mathsf{S}^2(\mathbb{R})$ is given by

$$F^* g_{\mathsf{S}^2(\mathbb{R})}(\theta,\varphi) \sim \begin{pmatrix} \sin^2\varphi & 0 \\ 0 & 1 \end{pmatrix}.$$

Definition [Vertical and horizonal subspaces, Riemannian submersions] Let $\pi : \widetilde{M} \to M$ be a surjective submersion. The fiber over $q \in M$, written F_y, is defined as the inverse image $F_q = \pi^{-1}(q) \subset \widetilde{M}$ (note: since π has constant rank, F_q is a closed, embedded submanifold of $\widetilde{M}$).

Suppose $(\widetilde{M}, \widetilde{g})$ is a Riemannian manifold. Let $p \in F_q$. The vertical space at p is the subspace of $T_p\widetilde{M}$ defined as

$$\mathsf{V}_p = \mathsf{Ker}\,\pi_*$$

where $\pi_* : T_p\widetilde{M} \to T_qM$ is the push-forward of the map π.

The horizontal subspace at p is defined as $\mathsf{H}_p = \mathsf{V}_p^\perp$ and corresponds to the orthogonal complement of V_p with respect to the inner-product $\widetilde{g}_p : T_p\widetilde{M} \times T_p\widetilde{M} \to \mathbb{R}$.

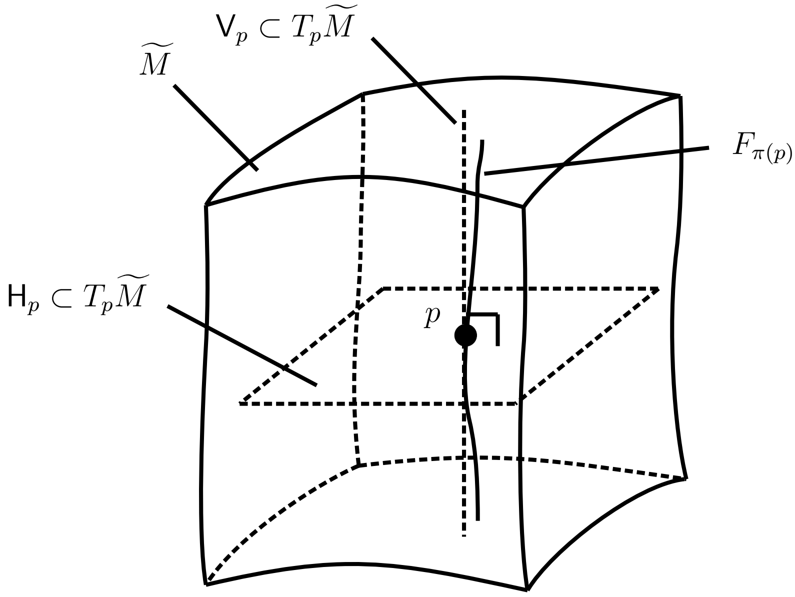
Thus, we have the orthogonal direct sum

$$T_p\widetilde{M} = \mathsf{V}_p \oplus \mathsf{H}_p.$$

Let g be a Riemannian metric on M. The map $\pi : \widetilde{M} \to M$ is said to be a Riemannian submersion if, for every $p \in \widetilde{M}$, the restriction

$$\pi_* : \mathsf{H}_p \to T_{\pi(p)}M$$

is an isometry.



Proposition [Horizontal lifts, Riemannian submersions] Let $\widetilde{M}$ be a Riemannian manifold and $\pi : \widetilde{M} \to M$ be a surjective submersion.

(a) Any smooth vector field $\widetilde{X}$ on $\widetilde{M}$ can be written uniquely as

$$\widetilde{X} = \widetilde{X}^{\mathsf{H}} + \widetilde{X}^{\mathsf{V}}$$

where $\widetilde{X}^{\mathsf{H}}$ and $\widetilde{X}^{\mathsf{V}}$ are smooth vector fields on $\widetilde{M}$ with $\widetilde{X}^{\mathsf{H}}_p \in \mathsf{H}_p$ and $\widetilde{X}^{\mathsf{V}}_p \in \mathsf{V}_p$ for all $p \in \widetilde{M}$.

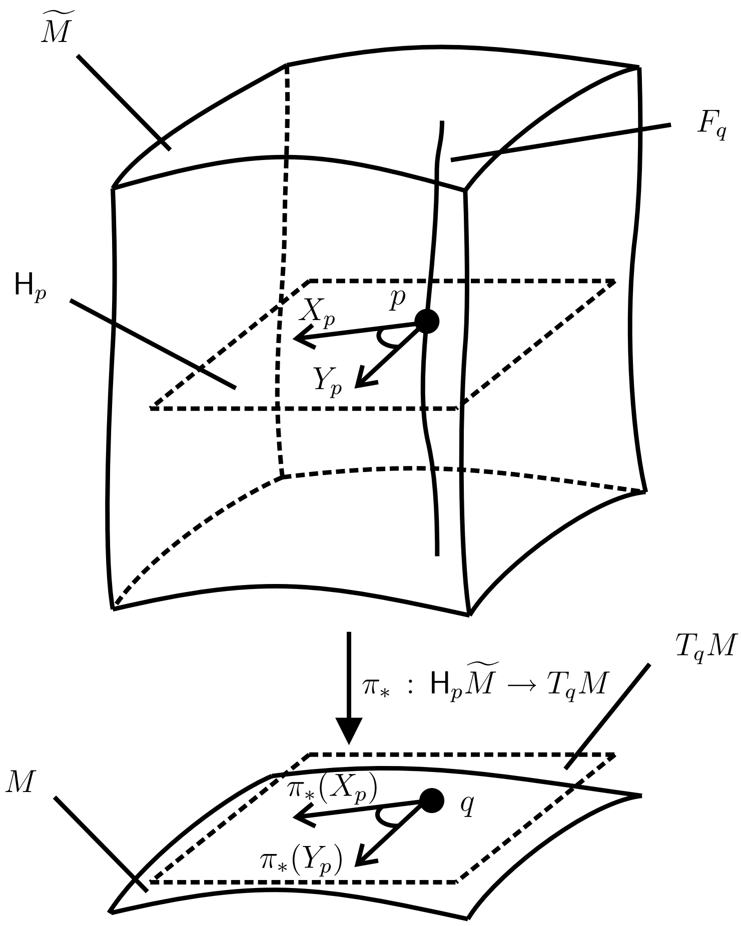
(b) Let X be a smooth vector field on M. Then, there is an unique horizontal smooth vector $\widetilde{X}^{\mathsf{H}}$ on $\widetilde{M}$, called the horizontal lift of X, such that

$$\pi_* \left(\widetilde{X}^{\mathsf{H}}_p \right) = X_{\pi(p)},$$

for all $p \in M$.

(c) Let φ denote a smooth action of a Lie group G on $\widetilde{M}$ such that:

- G preserves fibers (i.e, $\pi \circ \varphi_g = \pi$ for all $g \in G$)
- G acts transitively on each fiber



- G acts on $\widetilde{M}$ by isometries (i.e., each linear map $\varphi_{g*} : T_p\widetilde{M} \to T_{\varphi_g(p)}\widetilde{M}$ is an isometry).

Then, there exists a unique Riemannian metric on M such that π is a Riemannian submersion.

Example 1 **[A Riemannian metric on** $\mathsf{P}(n, \mathbb{R})$**]** Consider the map

$$\pi : \mathsf{GL}(n, \mathbb{R}) \to \mathsf{P}(n, \mathbb{R})$$

which, given $A \in \mathsf{GL}(n, \mathbb{R})$, extracts the P factor of A from the polar decomposition $A = PQ$.

Note that π is a surjective submersion.

The vertical space at $A \in \mathsf{GL}(n,\mathbb{R})$ is given by $\mathsf{V}_A = A\mathsf{K}(n,\mathbb{R})$.

Consider the previously discussed left-invariant metric on $\mathsf{GL}(n,\mathbb{R})$. Then, the horizontal space at $A \in \mathsf{GL}(n,\mathbb{R})$ is given by $\mathsf{H}_A = A\mathsf{S}(n,\mathbb{R})$.

The projection map

$$\pi_* : \mathsf{H}_A \simeq A\mathsf{S}(n,\mathbb{R}) \to T_P\mathsf{P}(n,\mathbb{R}) \simeq \mathsf{S}(n,\mathbb{R}) \qquad AS \to \Delta$$

is described by the equation

$$\Delta P + P\Delta = 2ASA^T.$$

The Lie group $\mathsf{O}(n)$ acts smoothly on $\mathsf{GL}(n,\mathbb{R})$ as

$$\mathsf{O}(n) \times \mathsf{GL}(n,\mathbb{R}) \to \mathsf{GL}(n,\mathbb{R}) \qquad Q \cdot A = AQ^T.$$

Note that $\mathsf{O}(n)$ preserves fibers, acts transitively on them and acts on $\mathsf{GL}(n,\mathbb{R})$ by isometries.

The Riemannian metric on $\mathsf{P}(n)$ that makes π a Riemannian submersion is given by

$$g_P\left(\Delta,\Omega\right) = \frac{1}{4}\,\mathsf{tr}\left(P^{-1}\Delta + \Delta P^{-1}\right)\left(P^{-1}\Omega + \Omega P^{-1}\right).$$

References

[1] J. Lee, *Riemannian Manifolds*, Springer-Verlag, 2000.