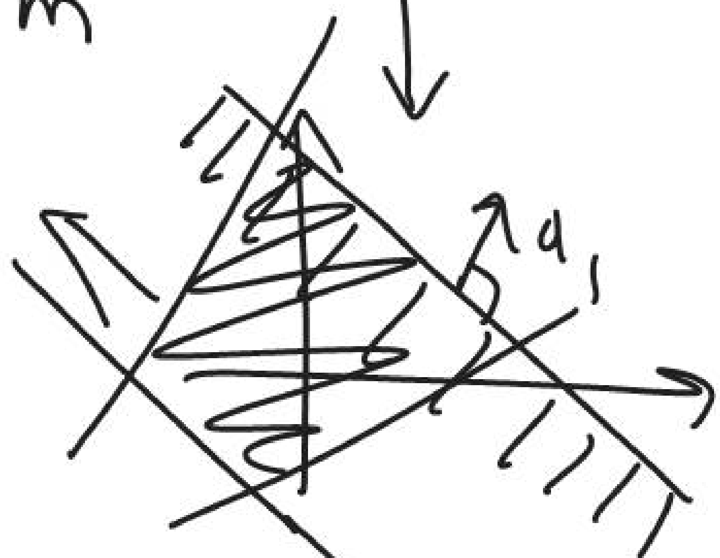
$$Ax \leq b$$

$$Ax \le b \iff \left\{ \begin{array}{l} a_1^T x \le b_1 \\ \vdots \\ a_m^T x \le b_m \end{array} \right.$$

$$A = \begin{bmatrix} - a_1^T - \\ \\ - a_m^T - \end{bmatrix} \qquad b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$$m \times n \qquad\qquad m$$



$$\|Ax\| \leq \sigma_{max}(A)\,\|x\|$$

$$x \longrightarrow \boxed{\;A\;} \longrightarrow Ax$$

$$\sigma_{max}(A) = \sup_{\substack{\|u\|=1 \\ \|v\|=1}} u^T A v$$

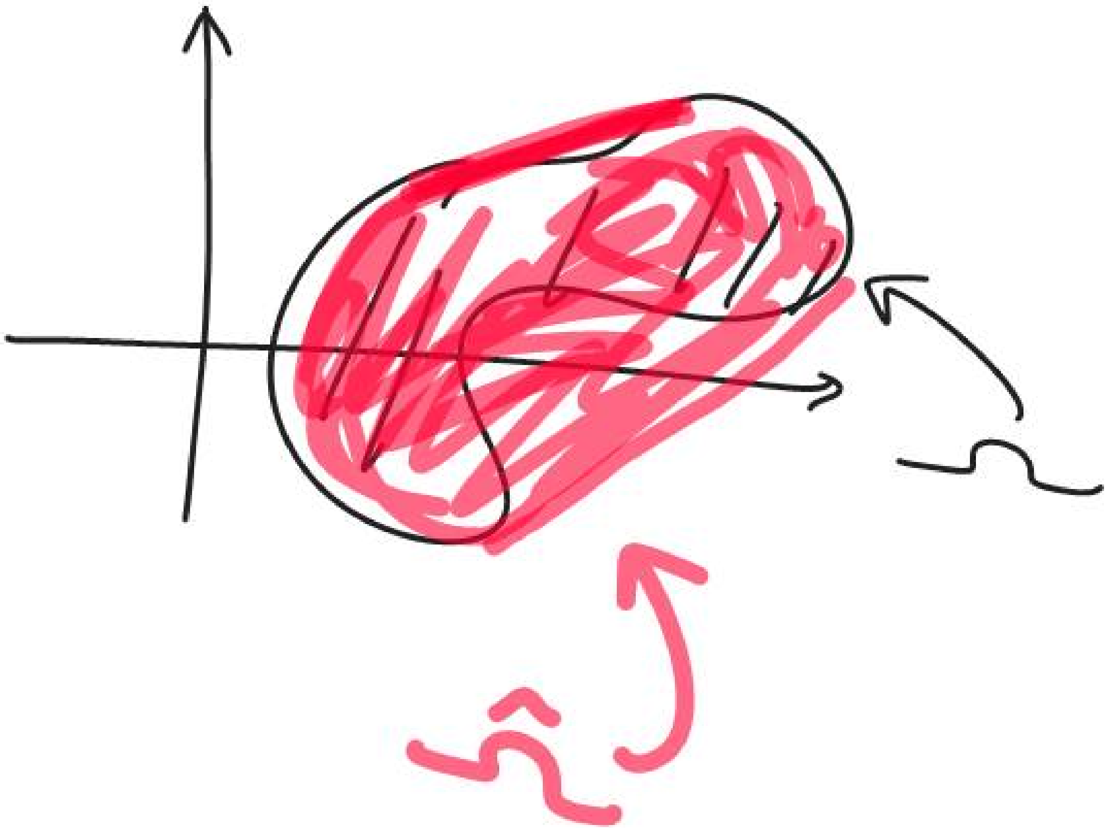
$$\sigma_{max}(A) \leq 1 \iff \forall_{\substack{\|u\|=1 \\ \|v\|=1}} : u^T A v \leq 1$$

$$\{A \in \mathbb{R}^{m \times n} : \sigma_{max}(A) \le 1\}$$

$$\|$$

$$\bigcap_{\substack{\|u\|=1 \\ \|v\|=1}} \underbrace{\{A \in \mathbb{R}^{m \times n} : u^T A v \le 1\}}_{S_{(u,v)}}$$

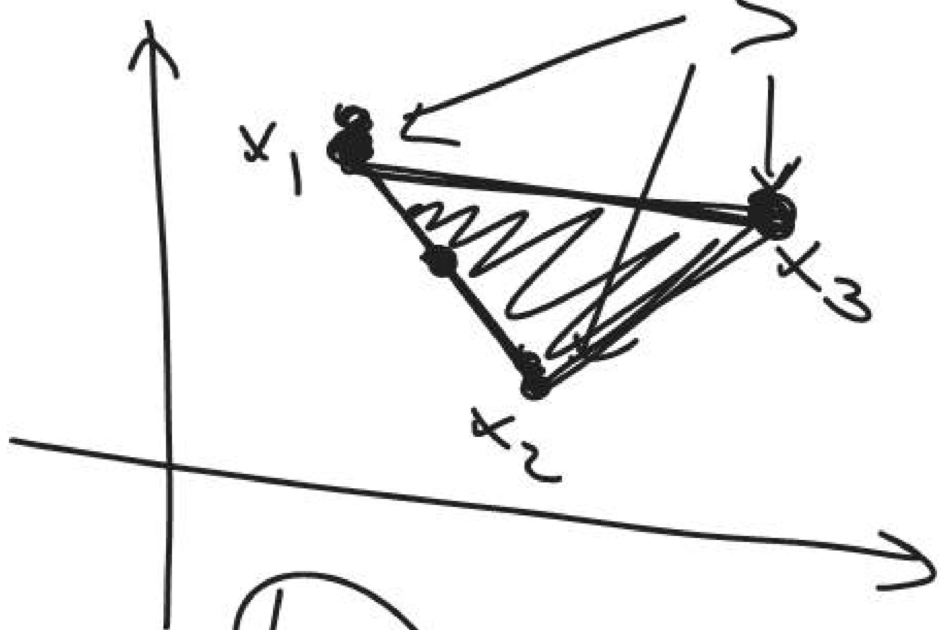
$$J = \{(u,v) : \|u\| = \|v\| = 1\} \qquad j = (u,v)$$



$$\min.\ f(x)$$

$$\text{s.t. } x \in \cancel{\Omega}\ \hat{\Omega}$$

$$\operatorname{co} S = \left\{ \sum_{i=1}^{k} \alpha_i x_i : x_i \in S,\ \alpha_i \geq 0,\ \alpha_1 + \dots + \alpha_k = 1,\ k = 1, 2, \dots \right\}$$



$k=3$

$$\alpha_1 x_1 + \alpha_2 x_2 + \alpha_3 x_3$$

$k=2$

$$\alpha_1 x_1 + \alpha_2 x_2$$

$\in S$

$(\alpha_1, \alpha_2) = (1, 0)$

$(\alpha_1, \alpha_2) = (1/2, 1/2)$

$$S = \{x \in \mathbb{R}^n : g_1(x) \le 0, \ldots, g_m(x) \le 0\}$$

$\downarrow$

$$\mathrm{co}\, S = ?$$

$$S_{(u,v)} = \{ A \in \mathbb{R}^{m \times n} : \quad u^T A v \le 1 \}$$

$$u^T A v = \operatorname{tr}(u^T A v) = \operatorname{tr}(v^T A^T u) = \boxed{\operatorname{tr}(A^T u v^T)} = \langle A, u v^T \rangle$$

$$\boxed{\begin{aligned} &\operatorname{tr}(Z) \\ &\quad \shortparallel \\ &\operatorname{tr}(Z^T) \end{aligned}}$$

$$\boxed{\operatorname{tr}(ABC) = \operatorname{tr}(BCA)} = \boxed{\operatorname{tr}(CAB)}$$

$$V = \mathbb{R}^n \qquad \langle u, v \rangle = u^T v$$

$$= u_1 v_1 + \cdots + u_n v_n$$

$$V = \mathbb{R}^{n \times m} \qquad \langle A, B \rangle = \operatorname{tr}(B^T A)$$

$$= \operatorname{tr}(A^T B)$$

$$= \sum_{ij} A_{ij} B_{ij}.$$

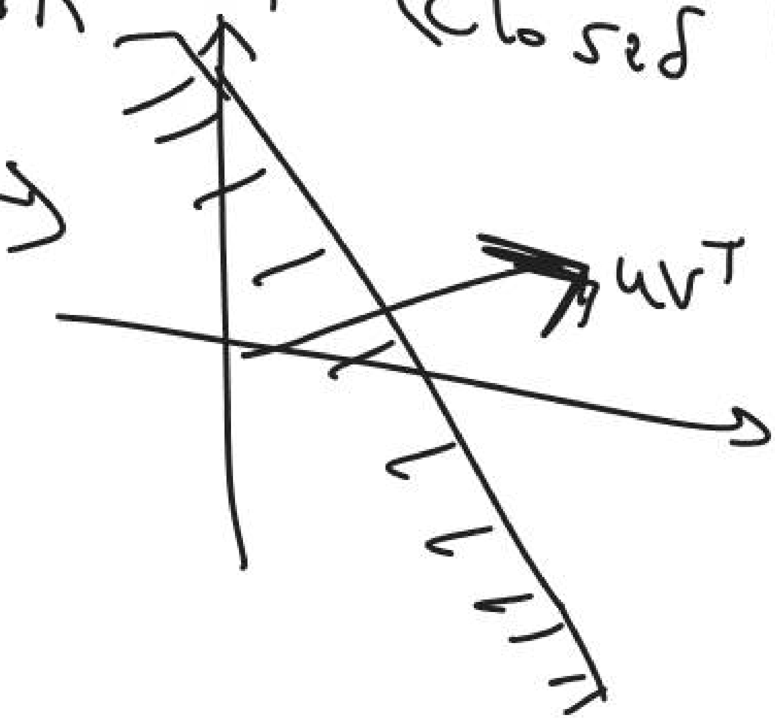
$$A \longrightarrow \begin{bmatrix} a_1 \\ \vdots \\ a_m \end{bmatrix} = \operatorname{vec}(A)$$

$$A = [a_1 \cdots a_m]$$

$$S_{(u,v)} = \{ A \in \mathbb{R}^{m \times n} : u^T A v \le 1 \}$$

$$= \{ A \in \mathbb{R}^{m \times n} : \langle A, uv^T \rangle \le 1 \}$$

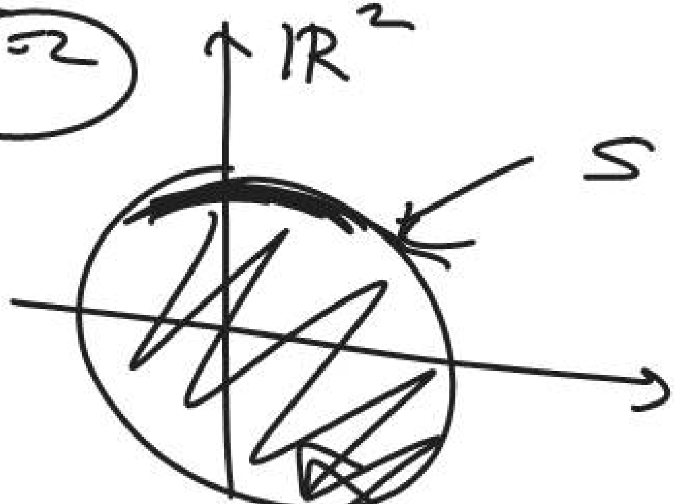
$\mathbb{R}^{m \times n}$ (Closed half-space)



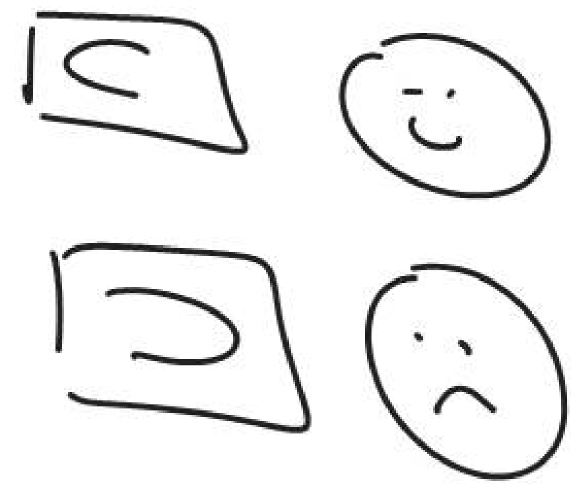
: cvx

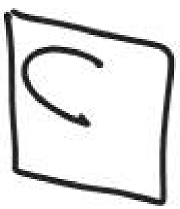
$$S = \{x \in \mathbb{R}^n : \|x\|_p = 1\}$$

$p=2$



$$\operatorname{co} S = \{x : \|x\|_p \le 1\}$$





1) Pick $x \in \operatorname{co} S$

$x = \alpha_1 x_1 + \dots + \alpha_k x_k \qquad \alpha_i \ge 0,\ \alpha_1 + \dots + \alpha_k = 1$

$x_i \in S$

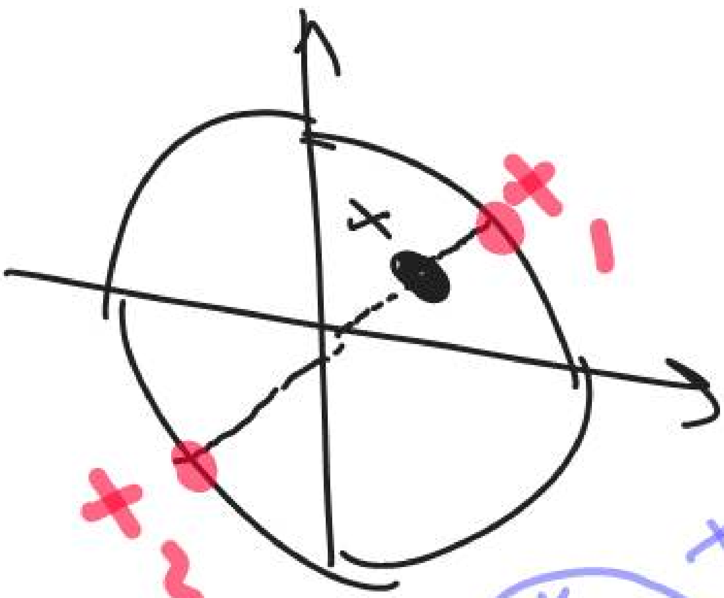
2) $\|x\|_p \le \alpha_1 \boxed{\|x_1\|_p} + \dots + \alpha_k \boxed{\|x_k\|_p} = 1$

$\boxed{?}$ 1) Pick x s.t. $\|x\|_p \le 1$

Goal: <u>find</u> ✓ $\alpha_1, \dots, \alpha_k \ge 0$ s.t. $x = \sum_{i=1}^{k} \alpha_i x_i$

✓ $\alpha_1 + \dots + \alpha_k = 1$

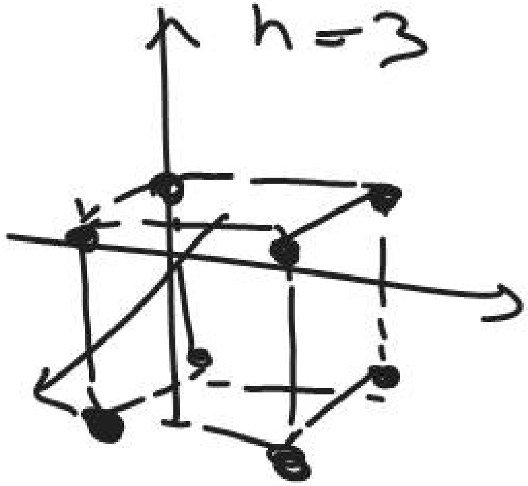
✓ $x_1, \dots, x_k \in S$



$$x = \overset{\alpha_1}{\boxed{\frac{1+\|x\|_p}{2}}} \left(\frac{x}{\|x\|_p} \right) + \overset{\alpha_2}{\boxed{\frac{1-\|x\|_p}{2}}} \left(-\frac{x}{\|x\|_p} \right)$$

$x \neq 0$

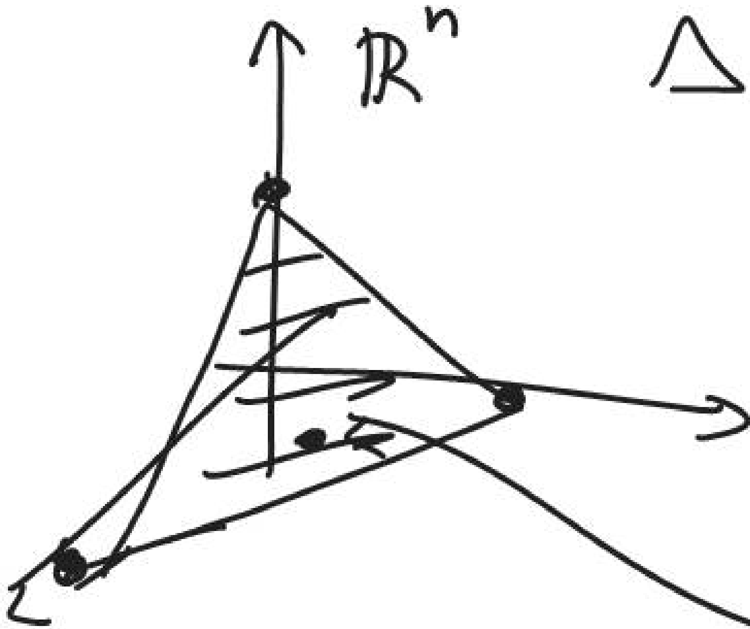
• $S = \{\underbrace{(\pm 1, \dots, \pm 1)}_{n}\}$ $\quad |S| = 2^n$



$\mathrm{co}\, S = \{x : \|x\|_\infty \le 1\}$

$\boxed{\subset}$

$\boxed{\supset}$



Δ_n

$X \sim \{a, b, c, d, e\}$

$$\begin{cases} \mathbb{P}(X = a) = p_1 \\ \quad \vdots \\ \mathbb{P}(X = e) = p_5 \end{cases}$$

$$p = \begin{bmatrix} p_1 \\ \vdots \\ p_5 \end{bmatrix}$$

$$S = \{(1,0,0), (0,1,0), (0,0,1)\}$$

$$\mathrm{co}\, S = \{x : x \geq 0, \mathbf{1}^T x = 1\} = \Delta_n$$

unit-sphere (n=1)

orthogonal matrices (m=n)

Stieffel

$$S = \{ X \in \mathbb{R}^{m \times n} : X^T X = I_n \}$$