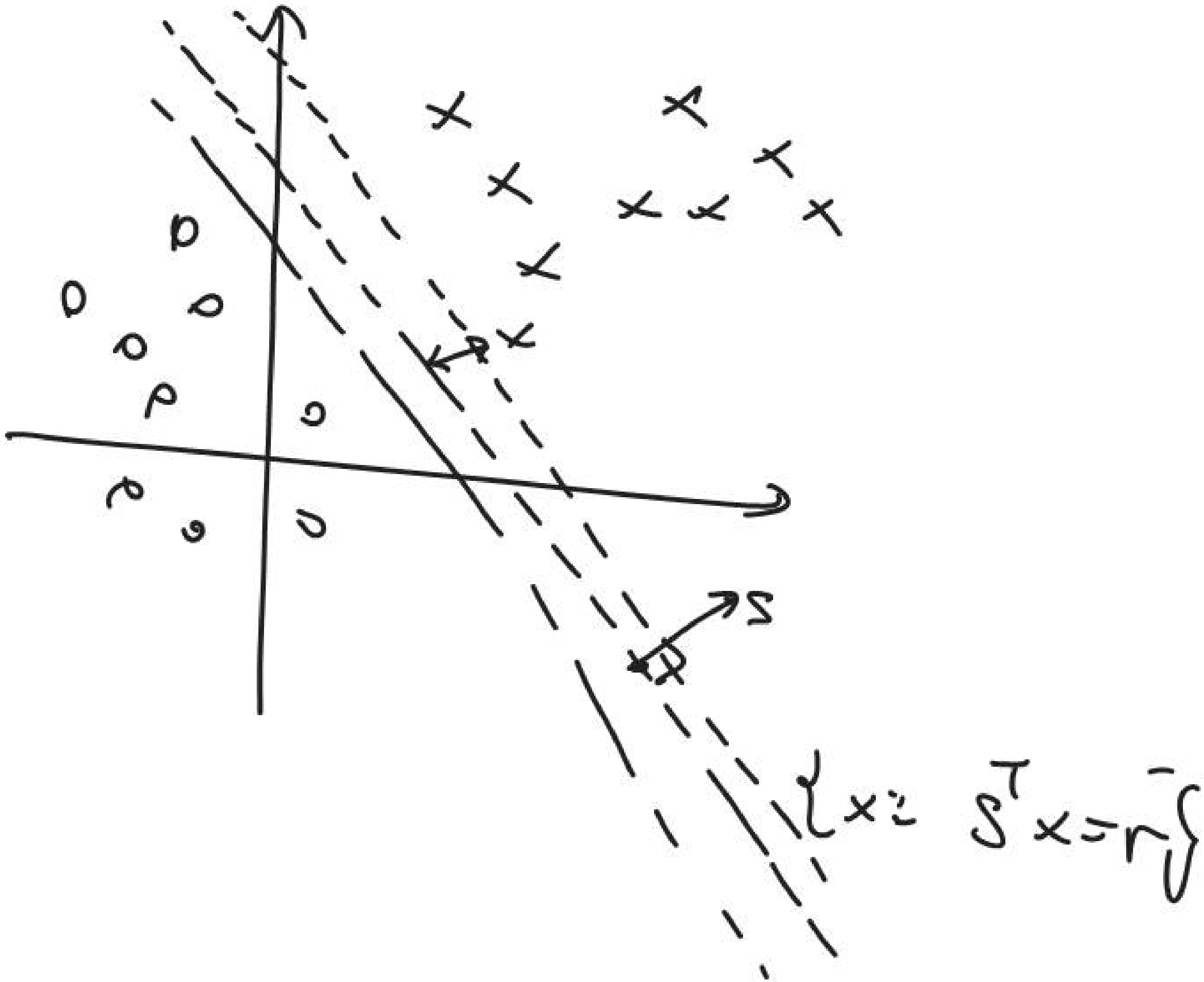

s
$\{x : s^T x = \bar{r}\}$

min. $\|s\|^2$

s.t. $X^T s \ge (r+1)\,1_k \leftarrow \mu \qquad (\dots \le 0)$

$Y^T s \le (r-1)\,1_L \leftarrow \nu \qquad (\dots \le 0)$

1) $L(s,r;\mu,\nu) = \|s\|^2 + \mu^T\big((r+1)1_k - X^T s\big) + \nu^T\big(Y^T s - (r-1)1_L\big)$

$= \boxed{\|s\|^2 - s^T(X\mu - Y\nu)} + \boxed{r\,(\mu^T 1_k - \nu^T 1_L)} + 1_k^T\mu + 1_L^T\nu$

$\nabla(\cdot)(s^*) = 0$

$2s^* - (X\mu - Y\nu) = 0$

$s^* = \frac{X\mu - Y\nu}{2}$

2) $L(\mu,\nu) = \inf_{(s,r)} L(s,r;\mu,\nu) = \begin{cases} -\infty; & 1_k^T\mu - 1_L^T\nu \neq 0 \\ -\frac{1}{4}\|X\mu - Y\nu\|^2; & 1_k^T\mu = 1_L^T\nu \end{cases}$

2) $$L(\mu,\nu) = \begin{cases} -\infty \; ; & 1_k^T \mu \neq 1_L^T \nu \\ -\dfrac{\|X\mu - Y\nu\|^2}{2} + 1_k^T\mu + 1_L^T \nu \; ; & \end{cases}$$

3) $$\max_{\substack{\mu \geq 0 \\ \nu \geq 0}} \; L(\mu,\nu)$$

$1_k^T \mu = 1_L^T$ if $1_k^T\mu = 1_L^T \nu$

$$= \max \; -\frac{\|X\mu - Y\nu\|^2}{2} + 1_k^T \mu + 1_L^T \nu$$

s.t.
$\mu \geq 0$
$\nu \geq 0$
$1_k^T \mu = 1_L^T \nu$

$1_k^T \mu \neq 1_L^T \nu$

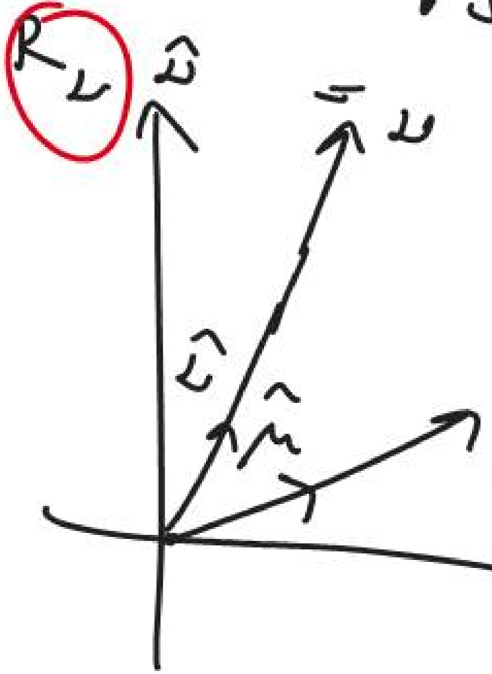
$R_\mu \qquad R_\nu$

max. $-\frac{1}{4}\|X\mu - Y\nu\|^2 + 1_k^T\mu + 1_L^T\nu$

s.t. $\mu \geq 0,\ \nu \geq 0,\ 1_k^T\mu = 1_L^T\nu$

$R_\mu \hat{\mu} \qquad R_\nu \hat{\nu}$

$$\left\| [X \ Y] \begin{bmatrix} \mu \\ -\nu \end{bmatrix} \right\|^2 = [\mu^T \ -\nu^T] \begin{bmatrix} X^TX & X^TY \\ Y^TX & Y^TY \end{bmatrix} \begin{bmatrix} \mu \\ -\nu \end{bmatrix}$$



$1^T\hat{\nu} = 1 \qquad R_\nu \geq 0$

$(\geq 0) \quad (\text{vector})$

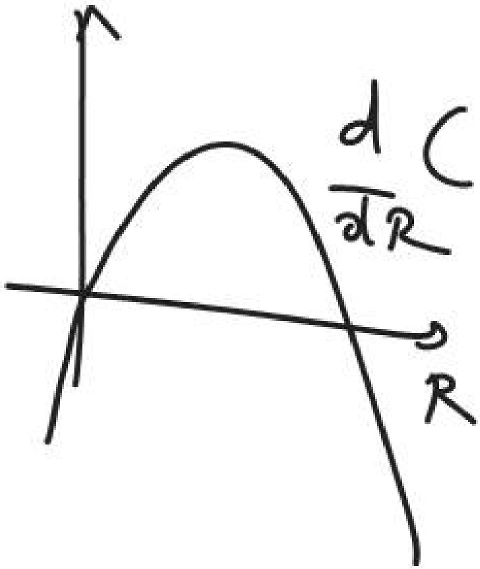
$\mu = R_\mu \hat{\mu}$

$1^T\hat{\mu} = 1$

"Kernel trick"

$$\max.\ -\frac{1}{4} R^2 \|X\hat{\mu} - Y\hat{\nu}\|^2 + 2R$$

$$\text{s.t.}\quad \hat{\mu}, \hat{\nu} \geq 0 \quad 1^T\hat{\mu} = 1^T\hat{\nu} = 1 \quad R \geq 0$$



$$\frac{d}{dR}(\ \ )(R^*) = 0$$

$$\downarrow$$

$$-\frac{1}{2} R^* \|X\hat{\mu} - Y\hat{\nu}\|^2 + 2 = 0$$

$$\downarrow$$

$$R^* = \frac{4}{\|X\hat{\mu} - Y\hat{\nu}\|^2}$$

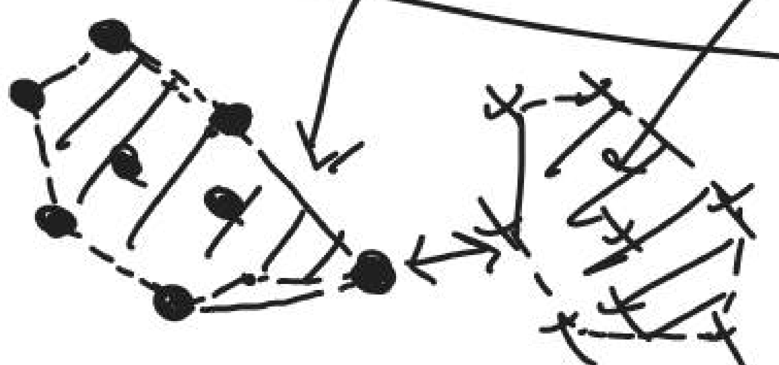
$$\boxed{\inf_{x,y} f(x,y) \quad = \quad \inf_{x} \underbrace{\left[\inf_{y} f(x,y)\right]}_{g(x)}}$$

$$\text{max.} \quad +4 \, \frac{1}{\|X\hat{\mu} - Y\hat{\nu}\|^2}$$

$$\text{s.t.} \quad 1^T\hat{\mu} = 1^T\hat{\nu} = 1 \quad \hat{\mu}, \hat{\nu} \geq 0$$

$$\text{min.} \quad \|X\hat{\mu} - Y\hat{\nu}\|^2$$

$$\text{s.t.} \quad \hat{\mu}, \hat{\nu} \geq 0 \quad 1^T\hat{\mu} = 1^T\hat{\nu} = 1$$



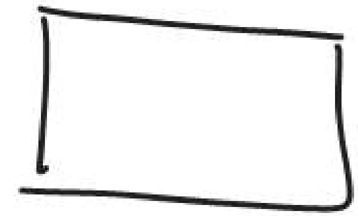
min. $f(x)$

s.t.

$\longrightarrow$

min. $g(x, y)$

s.t.



$$\max. \quad r^T x$$

$$\text{s.t.} \quad x \geq 0, \quad 1^T x = T$$

$$x_{[1]} \cancel{+ x_{[2]}} \leq 0.8\,T$$

1 $\begin{array}{l} \min.\ c^T x \\ \text{s.t. } Ax \leq b \end{array}$

$$\begin{aligned}
x_1 + x_2 &\leq 0.8T \\
x_1 + x_3 &\leq 0.8T \\
&\vdots \\
x_1 + x_n &\leq 0.8T \\
&\vdots \\
x_{n-1} + x_n &\leq 0.8T
\end{aligned}$$

$$\binom{n}{k} \sim n^k$$

Bender's decomposition

Dantzig-Wolfe

$$\lambda_1(x) + \cdots + \lambda_k(x) \leq \alpha$$

$$\lambda(x) = \begin{bmatrix} \lambda_1(x) \\ \vdots \\ \lambda_n(x) \end{bmatrix}$$

Ky Fan

$$(x_{[1]} + \cdots + x_{[h]}) \le 0.8T$$

$$x_{[1]} + \cdots + x_{[h]} = \boxed{\begin{array}{ll} \max. & f(x,y) \\ \text{s.t.} & y \in Y \end{array}}$$

$\downarrow$ Duality (solvable)

$$\boxed{\begin{array}{ll} \min. & g(x,z) \\ \text{s.t.} & h(x,z) \le 0 \end{array}} \le 0.8T$$

$$\boxed{\begin{array}{l} g(x,z) \le 0.8T \\ h(x,z) \le 0 \end{array}}$$

$$\left[\begin{array}{ll} \min. & \phi(z) \\ \text{s.t.} & z \in Z_1 \end{array}\right] \leq \eta$$

solvable

$$\Uparrow \qquad \Updownarrow \qquad \Downarrow$$

solvable $\Longrightarrow \exists\ z^* \in Z : \phi(z^*) \leq \eta$

$$\left[\begin{array}{ll} \min. & \phi(z) \\ \text{s.t.} & z \in Z \end{array}\right] = \phi(z^*)$$

$$\Downarrow$$

$$\begin{cases} \phi(z^*) \leq \eta \\ z^* \in Z_1 \end{cases}$$

$$\begin{cases} \phi(z) \leq \eta \\ z \in Z_1 \end{cases}$$

$$z_0 \in Z_1,\ \phi(z_0) \leq \eta \Longrightarrow \left[\begin{array}{ll} \inf & \phi(z) \\ \text{s.t.} & z \in Z_1 \end{array}\right] \leq \eta$$

$$\begin{aligned} \min.\ & \phi(z) \\ \text{s.t. } & z \in Z \end{aligned}$$

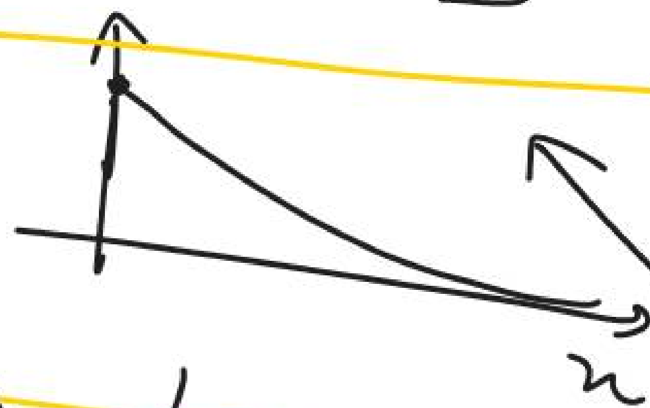
solvable

$$\Updownarrow$$

$$\exists\, z^* \in Z : \phi(z^*) = \inf\{\phi(z) : z \in Z\}$$

$$= \left[\begin{aligned} \min.\ & \phi(z) \\ \text{s.t. } & z \in Z \end{aligned}\right]$$

$$\boxed{\begin{bmatrix} \min. & e^{-x} \\ \text{s.t.} & x \geq 0 \end{bmatrix} \leq 0}$$



not solvable!

$$\boxed{\begin{matrix} \cancel{e^{-x}} \leq 0 \\ \cancel{x \geq 0} \end{matrix}}$$