SDP

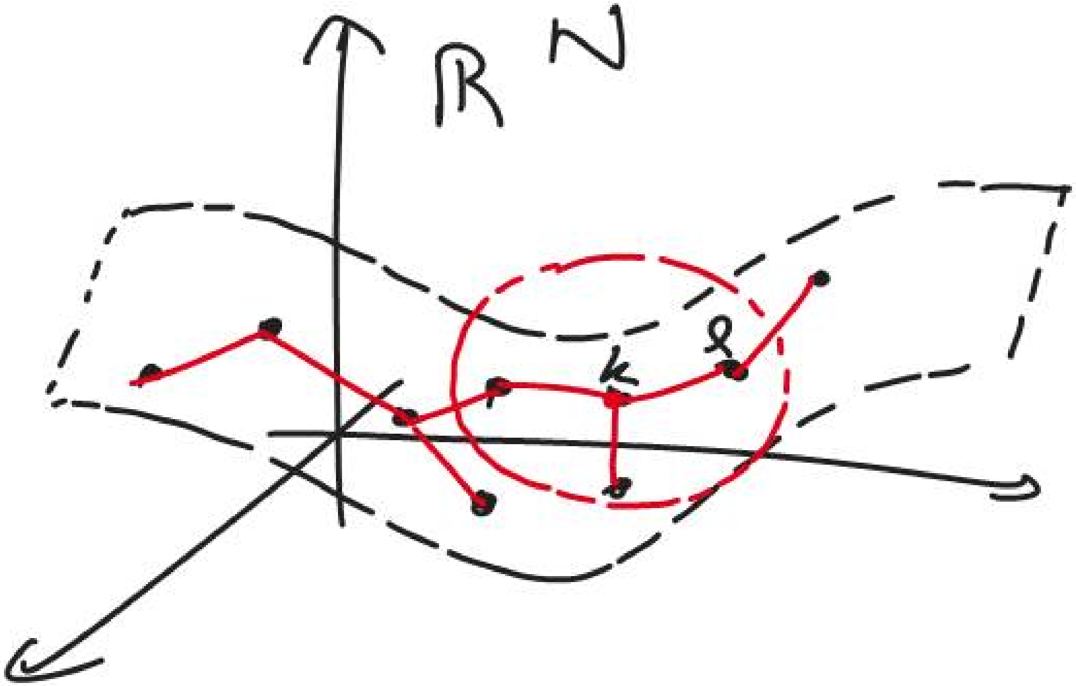
$$x = \begin{bmatrix} x_1 \\ \vdots \\ x_n \end{bmatrix}$$

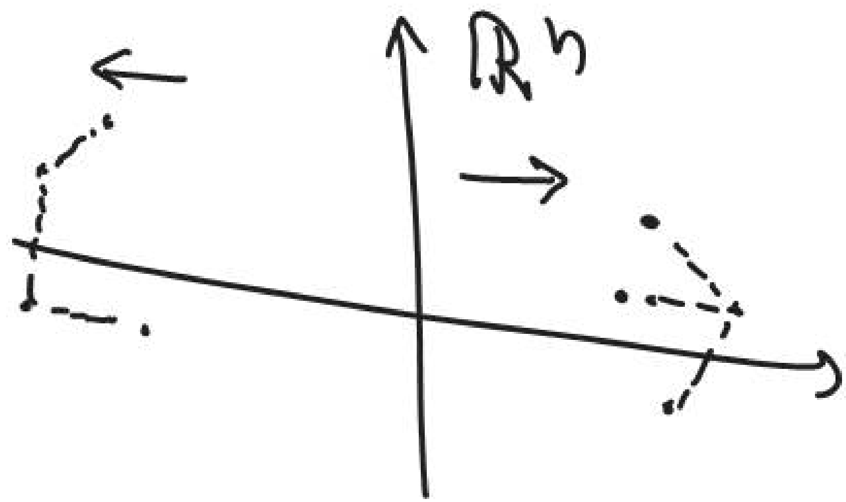
$$\min. \quad c^T x$$

$$\text{s.t.} \quad \underbrace{A_0 + x_1 A_1 + \cdots + x_n A_n}_{\in S^n_x} \succeq 0$$

$$B_0 + x_1 B_1 + \cdots + x_n B_n \succeq 0$$

$$\left(A_i, B_i \in S^n \right)$$


$\mathbb{R}^N$
k
l


$\mathbb{R}^n$

$$Y = [y_1\ y_2\ \cdots\ y_K] \in \mathbb{R}^{n \times K}$$

$(n << K)$

$$\sim 1 \sim \qquad K \left[\begin{array}{c} 0 \\ \vdots \\ 1 \\ \vdots \\ 0 \end{array}\right] \leftarrow k$$

$$\sum_k \|y_k\|^2 = \sum_k y_k^T y_k = \sum_k (Y e_k)^T (Y e_k)$$

$$= \sum_k e_k^T \, Y^T Y \, e_k$$

Gram Matrix

$= tr(Y^T Y)$

$$Y^T Y = \begin{bmatrix} y_1^T \\ \vdots \\ y_k^T \\ \vdots \\ y_K^T \end{bmatrix} \begin{bmatrix} y_1 & \cdots & y_l & \cdots & y_K \end{bmatrix}$$

$$= \begin{array}{cc} & \begin{array}{ccc} & l & \end{array} \\ k & \begin{bmatrix} & \vdots & \\ \cdots & y_k^T y_l & \cdots \\ & \vdots & \end{bmatrix} \end{array} : K \times K$$

$$\|y_h - y_e\|^2 = y_h^T y_h - 2 y_h^T y_e + y_e^T y_e$$

$$e_h^T Y^T Y e_h - 2 e_h^T Y^T Y e_e + e_e^T Y^T Y e_e$$

$$\begin{bmatrix} \text{min.} & x \\ \text{s.t.} & x \geq y^2 \end{bmatrix} \longrightarrow \begin{bmatrix} \text{min.} & x \\ \text{s.t.} & x \in \mathbb{R} \end{bmatrix} = -\infty$$

$= 0$

var: (x,y)

$$\begin{bmatrix} \text{min.} & x \\ \text{s.t.} & x \geq 0 \end{bmatrix} = 0$$

$$G = \underbrace{\begin{bmatrix} \\ Y^T \\ \\ \end{bmatrix}}_{n} \overbrace{\begin{bmatrix} Y \end{bmatrix}}^{K} \; \Big\updownarrow n$$

$K \times K$

$\exists \; Y : n \times K \; ! \; G = Y^T Y$

$\Updownarrow$

G : symm. ; $G \succeq 0$

rank $G \le n$

$x^T G x \ge 0, \; \forall_x$

$x^T Y^T Y x = \|Yx\|^2 \ge 0$

min.

s.t. $\qquad x^T y + \cdots \le 0$

$$z = \begin{bmatrix} x \\ y \end{bmatrix} \qquad \frac{1}{2} z^T \underbrace{\begin{bmatrix} 0 & I \\ I & 0 \end{bmatrix}}_{A} z = \frac{1}{2} \operatorname{tr}(A\, z z^T) = \frac{1}{2} \operatorname{tr}(A Z)$$

$$Z \succeq 0$$

$$\operatorname{rank} Z \le 1 \iff Z = z z^T$$

$$\begin{bmatrix} z \end{bmatrix} \begin{bmatrix} z^T \end{bmatrix}$$

min. $\quad \Phi(Z)$

s.t. $\quad \frac{1}{2} \operatorname{tr}(A Z) + \cdots \le 0$

$$\min \; f(x)$$
$$\text{s.t.} \; g(x) \le 0$$
$$x \ge 0$$

~~rank $X \le 1$~~

$x \ge 0$

~~rank $X \le 1$~~

Shor's relaxation

$$X = x x^T \implies X \succeq 0$$

~~rank $X \le 1$~~

"Recipe for quadratic problems"

max. $x^T A x$

s.t. $x^T A_i x \le c_i$ $\quad i=1,\dots,m$

$\downarrow$ S. Zhang

T. Luo, IEEE Sig. Proc. Mag.,

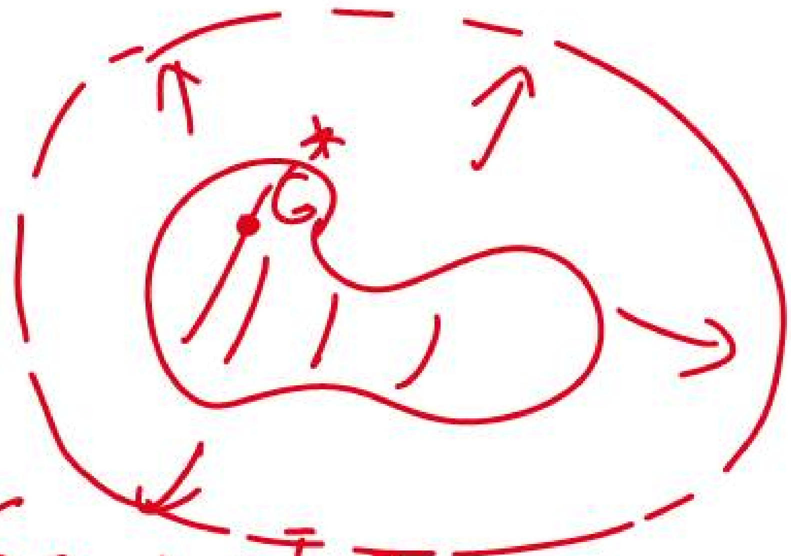
"Convex ..."

$$\boxed{\begin{aligned} \text{max.} \quad & \operatorname{tr}(AX) \\ \text{s.t.} \quad & \operatorname{tr}(A_i X) \le c_i \\ & X \succeq 0 \end{aligned}}$$

SDP

$\rightarrow$ $\cancel{\operatorname{tr}(X) \le 1}$

max. $\operatorname{tr}(G)$



s.t. $1^T G 1 = 0$

$$e_k^T G e_k - 2 e_k^T G e_\ell + e_\ell^T G e_\ell = \|x_k - x_\ell\|^2 \quad \forall (k,\ell) \in N$$

$$G \succeq 0$$

$$\downarrow$$

$$G^* = \begin{bmatrix} Y^T \end{bmatrix} \begin{bmatrix} Y \end{bmatrix}$$

if rank $G^* \leq n$

EVD

$$\underset{K \times K}{G^*} = \begin{array}{cc} \overset{n}{\longleftrightarrow} & \overset{K-n}{\longleftrightarrow} \\ \end{array}$$

$$G^* = \left[\, Q_1 \;\middle|\; Q_2 \,\right] \begin{bmatrix} \underset{n \times n}{\Lambda_1} & \\ & \underset{(K-n) \times (K-n)}{\Lambda_2} \end{bmatrix} \begin{bmatrix} Q_1^T \\ \hline Q_2^T \end{bmatrix}$$

$$\hat{G} = \left[\, Q_1 \;\middle|\; Q_2 \,\right] \begin{bmatrix} \Lambda_1 & \\ & 0 \end{bmatrix} \begin{bmatrix} Q_1^T \\ \hline Q_2^T \end{bmatrix} = \underbrace{\underset{K \times n}{\left(Q_1 \Lambda_1^{1/2}\right)} \left(Q_1 \Lambda_1^{1/2}\right)^T}$$

$Y^T = Q_1 \Lambda_1^{1/2}$

$Y^T Y$