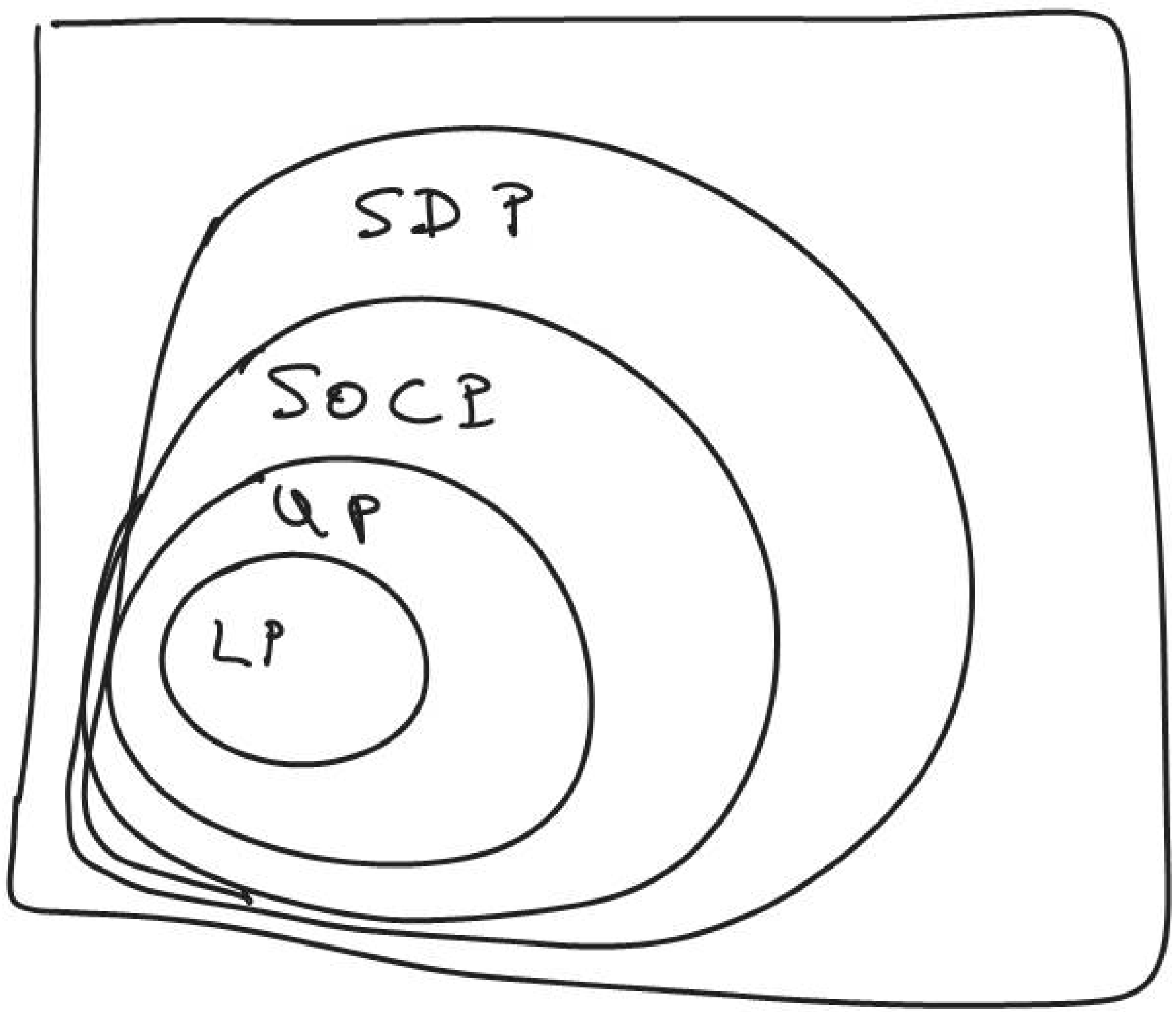
No class:

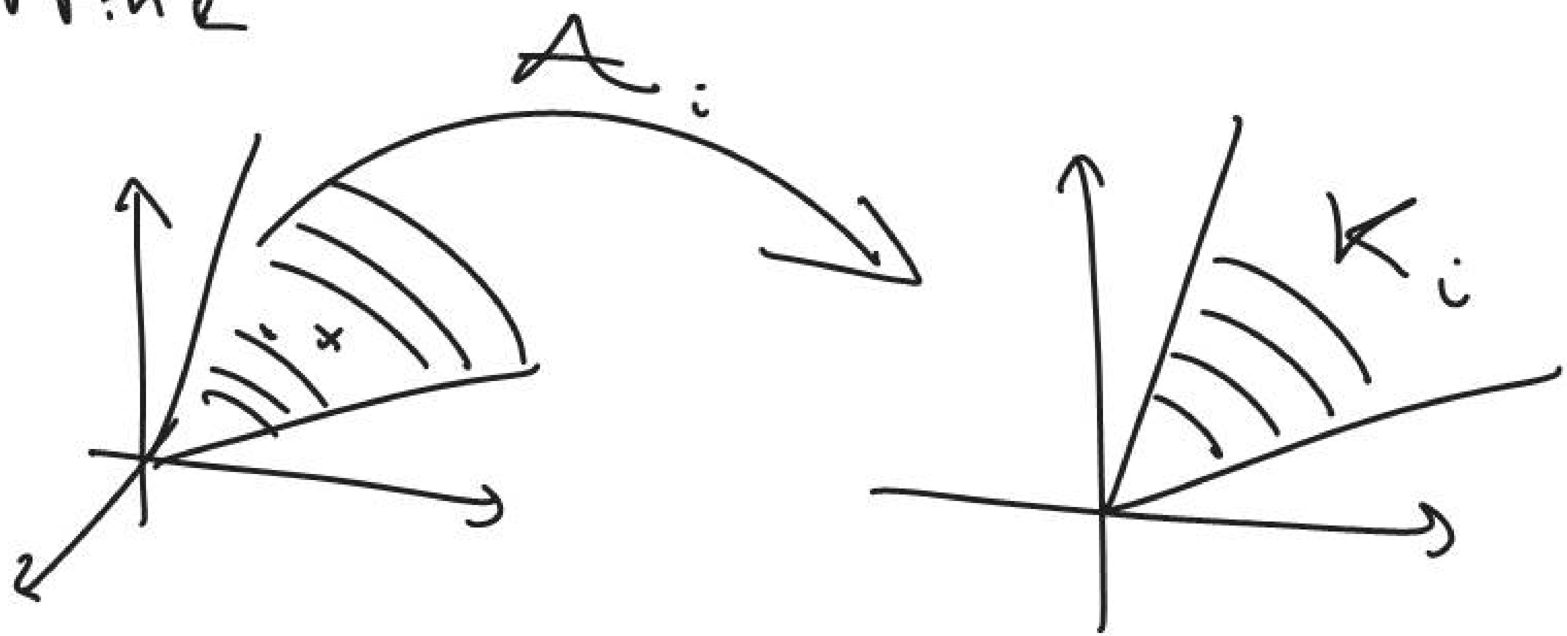
- March 28
- April 2


SDP
SOCP
QP
LP

$$A_i(x) \in K_i \iff x \in A_i^{-1}(K_i)$$

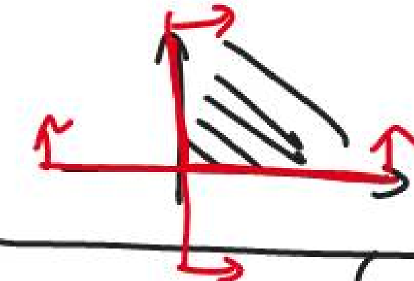
affine

convex cone



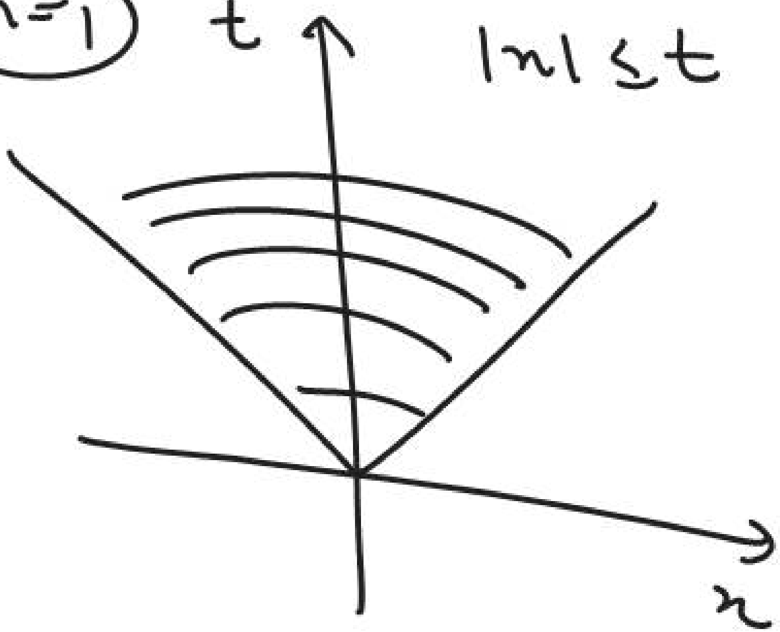
Second-Order Cone

n=2 $\mathbb{R}^2_+$

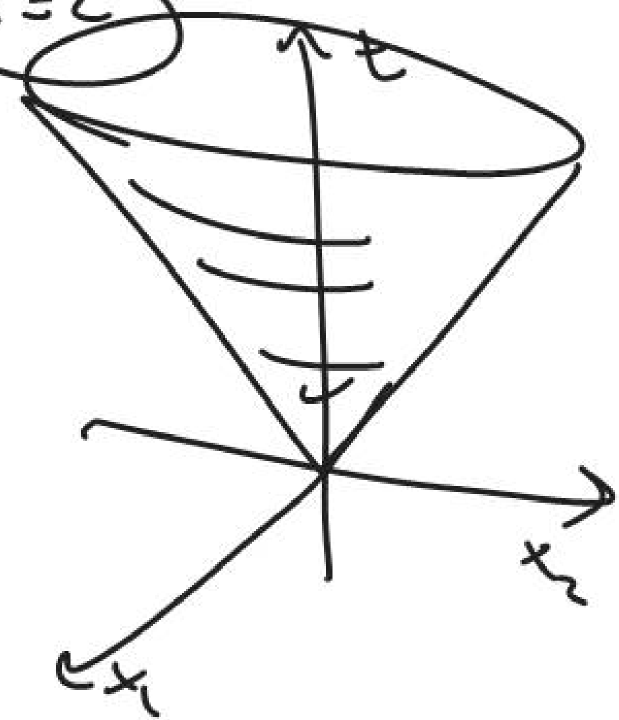


$$K = \{ (x,t) \in \mathbb{R}^{n+1} : \|x\| \le t \} \subset \mathbb{R}^{n+1}$$

n=1 $|x| \le t$



n=2



$$A_i(x) \in SOC \qquad \|C_i x + e_i\| \leq d_i^T x + f_i$$

$$\shortparallel$$

$$\begin{bmatrix} A_i \end{bmatrix} x + \begin{bmatrix} b \end{bmatrix} = \underbrace{\begin{bmatrix} C_i \\ \hline d_i^T \end{bmatrix}}_{A_i} x + \begin{bmatrix} e_i \\ \hline f_i \end{bmatrix} = \begin{bmatrix} C_i x + e_i \\ \hline d_i^T x + f_i \end{bmatrix}$$

$$\begin{array}{ll} \min. & \|Ax-b\|_\infty \\ \text{s.t.} & x \in \mathbb{C}^n \end{array} \longrightarrow \begin{array}{ll} \min. & t \\ \text{s.t.} & \|Ax-b\|_\infty \le t \end{array}$$

var. (x, t), $x \in \mathbb{C}^n$, $t \in \mathbb{R}$

$$A = \begin{bmatrix} a_1^T \\ \vdots \\ a_m^T \end{bmatrix} \quad b = \begin{bmatrix} b_1 \\ \vdots \\ b_m \end{bmatrix}$$

$$|z| = \sqrt{\mathrm{Re}\, z^2 + \mathrm{Im}\, z^2}$$

$$\max\{|a_1^T x - b_1|, \dots, |a_m^T x - b_m|\}$$

$$\begin{array}{ll} \min. & t \\ \text{s.t.} & |a_1^T x - b_1| \le t \\ & \vdots \\ & |a_m^T x - b_m| \le t \end{array}$$

$$|a_1^T x - b_1| \le t \longrightarrow \cancel{-t \le a_1^T x - b_1 \le t}$$

$$\begin{aligned} \min.\ & t \\ \text{s.t.}\ & |a_i^T x - b_i| \le t \qquad i=1,\dots,m \end{aligned}$$

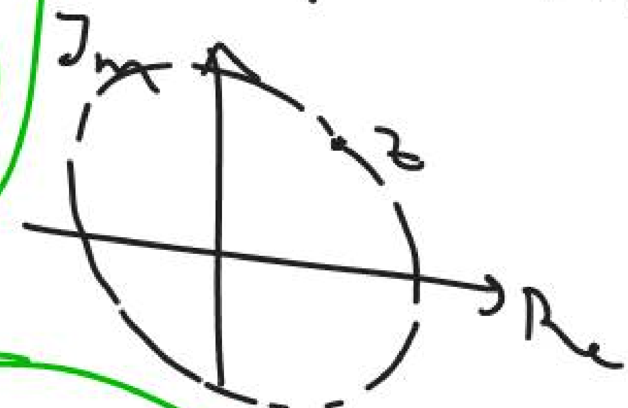
$$x \in \mathbb{C}^n, \quad t \in \mathbb{R}$$

$$\underset{\mathbb{C}^n}{x} = \underset{\mathbb{R}^n}{y} + j \underset{\mathbb{R}^n}{z}$$

$$b_i = c_i + j d_i$$

$$\underset{\mathbb{C}^n}{a_i} = \underset{\mathbb{R}^n}{\alpha_i} + j \underset{\mathbb{R}^n}{\beta_i}$$

$$|z| = \sqrt{\mathrm{Re}\, z^2 + \mathrm{Im}\, z^2}$$



$$= \left\| \begin{bmatrix} \mathrm{Re}\, z \\ \mathrm{Im}\, z \end{bmatrix} \right\|$$

$$|a_i^T x - b_i| = \left\| \begin{bmatrix} \alpha_i^T y - \beta_i^T z - c_i \\ \beta_i^T y + \alpha_i^T z - d_i \end{bmatrix} \right\|$$

Re

Im

$a_i = \alpha_i + j\beta_i$

$x = y + jz$

$b_i = c_i + jd_i$

$$\min \quad t$$

$$\text{s.t.} \quad \left\| \begin{bmatrix} \alpha_i^T & -\beta_i^T \\ \beta_i^T & \alpha_i^T \end{bmatrix} \begin{bmatrix} y \\ z \end{bmatrix} - \begin{bmatrix} c_i \\ d_i \end{bmatrix} \right\| \leq t \qquad i = 1, \dots, m$$

$$(t, y, z) \in \mathbb{R} \times \mathbb{R}^n \times \mathbb{R}^n$$

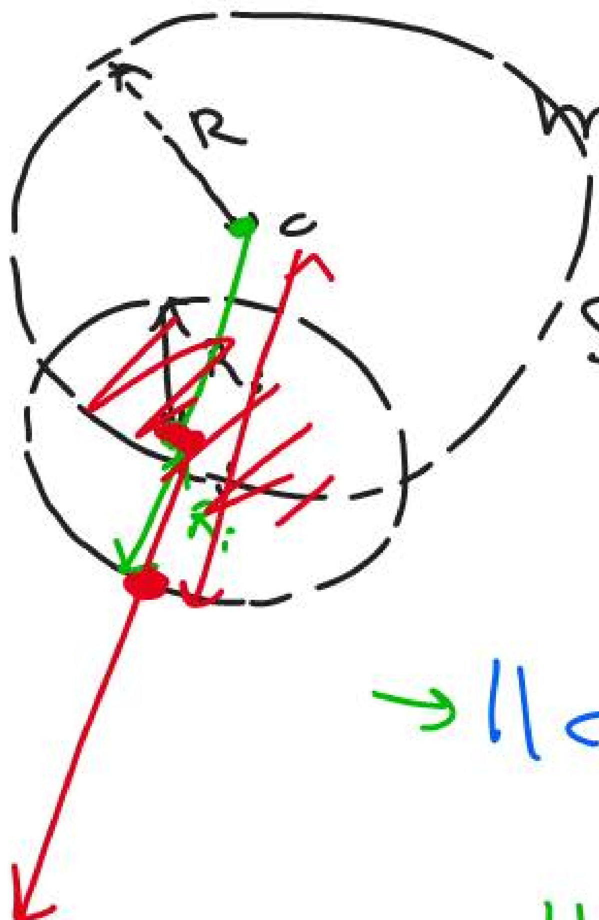
$$E_i \begin{bmatrix} t \\ y \\ z \end{bmatrix} + f_i$$

$$E_i = \left[\begin{array}{c|cc} 0 & \alpha_i^T & -\beta_i^T \\ & \beta_i^T & \alpha_i^T \end{array}\right]$$

$$t = g_i^T \begin{bmatrix} t \\ y \\ z \end{bmatrix} + h_i$$

$$\min.\ f(x)$$
$$\text{s.t.}\quad h_i(x) = 0$$
$$g_i(x) \le 0$$

Vor: (c, R)



$$\min.\ R$$

$$\text{s.t.}\quad \boxed{B(c_i, R_i) \subset B(c, R)}\ \forall i$$

$$\Updownarrow$$

$$\to \|c - R\mathbb{1}\| \ge \|c_i - R_i \mathbb{1}\|$$

$$\|c - c_i\| + R_i \le R$$

min. R

s.t. $\|c - c_i\| \le R - R_i \quad \forall$
$i = 1, \dots, m$

(c, R)

$$\left\| A_i \begin{bmatrix} c \\ R \end{bmatrix} + b_i \right\| \le e_i^T \begin{bmatrix} c \\ R \end{bmatrix} + f_i$$

$$A_i = \begin{bmatrix} I & 0 \end{bmatrix}$$

$$b_i = -c_i \qquad e_i = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$f_i = -R_i$$

$$\begin{cases} \|w\|^2 \leq yz \\ y \geq 0 \\ z \geq 0 \end{cases} \iff \left\| \begin{bmatrix} 2w \\ y - z \end{bmatrix} \right\| \leq y + z$$

$$\boxed{\Longrightarrow} \quad \left\| \begin{bmatrix} 2w \\ y - z \end{bmatrix} \right\|^2 = 4\|w\|^2 + (y-z)^2 \overset{?}{\leq} (y+z)^2$$

$$\Updownarrow$$

$$4\|w\|^2 - 2yz \leq 2yz$$

$$\Updownarrow$$

$$\cancel{4}\|w\|^2 \leq \cancel{4}yz$$

$$\boxed{\Leftarrow} \qquad \left\| \begin{bmatrix} 2w \\ \hline y-z \end{bmatrix} \right\| \le y+z \implies z \geq 0$$

$$\Downarrow$$

$$\sqrt{\underbrace{4\|w\|^2}_{\geq 0} + (y-z)^2} = y+z$$

$$\Downarrow$$

$$|y-z| = \sqrt{(y-z)^2} \le y+z$$

$$\Downarrow$$

$$\cancel{y} - z \le \cancel{y} + z \implies z \geq 0$$

$$\begin{aligned} \min.\quad & s + t \\ \text{s.t.}\quad & \|Ax - b\|^4 \le s \\ & \|\underbrace{Cx - d}_{w}\|^2 \le \underbrace{t}_{y}\ \underbrace{1}_{z} \end{aligned}$$

$$\begin{aligned} & \|w\|^2 \le y z \\ & y \ge 0,\ z \ge 0 \\ \hline & \left\| \begin{bmatrix} 2w \\ y - z \end{bmatrix} \right\| \le y + z \end{aligned}$$

$$\|Cx - d\|^2 \le t \iff \boxed{\left\| \begin{bmatrix} 2(Cx - d) \\ t - 1 \end{bmatrix} \right\| \le t + 1}$$

$$\|Ax-b\|^4 \le s \iff \begin{cases} u^2 \le s \\ \|\underbrace{Ax-b}_{w}\|^2 \le \underbrace{u}_{y} \end{cases}$$

$$\left\| \begin{bmatrix} 2u \\ s-1 \end{bmatrix} \right\| \le s+1$$

$$\left\| \begin{bmatrix} 2(Ax-b) \\ u-1 \end{bmatrix} \right\| \le u+1$$

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