Nonlinear Optimization (18799 B, PP)
IST-CMU PhD course
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Important: The homework is due April, 19.

Homework 4

Instructions: read section 4.6 of [1]. Solve **one** problem from the list below. If you solve two, we will pick the one with best score.

Problem A. *(Optimal covariance design)* Consider the problem of estimating the parameter $\theta \in \mathbb{R}^m$ from the measurement $y = A\theta + w$, where $A \in \mathbb{R}^{n \times m}$ is a full column-rank matrix, and $w \sim \mathcal{N}(0, R)$ denotes zero-mean Gaussian noise. The matrix A and the noise covariance matrix R (assumed positive definite) are known. The minimum variance estimator is given by $\widehat{\theta} = (A^\top R^{-1} A)^{-1} A^\top R^{-1} y$, and leads to the mean-square error

$$\mathrm{MSE}(R) = \mathbb{E}\left(\left\| \widehat{\theta} - \theta \right\|^2 \right) = \mathrm{tr}\left(\left(A^\top R^{-1} A \right)^{-1} \right).$$

Now, suppose that y is the output of a channel activated by a malicious eavesdropper that is trying to discover your "message" θ. (That is, there is also a legitimate channel, say, $z = B\theta + v$ with $v \sim \mathcal{N}(0, S)$, but this is of no concern to us in this problem.) Assume that you can tweak some system parameters and control the covariance R; more precisely, assume that you can choose R from the set

$$\mathcal{R} = \left\{ R \in \mathbb{R}^{n \times n} \,:\, \|r_i - c_i\| \le \rho_i, \text{ for } i = 1, \ldots, n \right\}.$$

Here, $r_i \in \mathbb{R}^i$ denotes the ith truncated column of R,

$$r_i = \begin{bmatrix} R_{1i} \\ R_{2i} \\ \vdots \\ R_{ii} \end{bmatrix}, \quad \text{for } i = 1, \ldots, n.$$

The vectors $c_i \in \mathbb{R}^i$ and the scalars $\rho_i > 0$ are given. The symbol R_{ij} denotes the (i,j)th entry of R. In words, you can control the diagonal of R and all entries above it (hence, by symmetry, the whole R). Assume that all $R \in \mathcal{R}$ are positive-definite matrices.

We want to find the covariance matrix R from the set $\mathcal{R}$ that maximizes the mean-square error of the eavesdropper,

$$\begin{array}{ll} \text{maximize} & \mathrm{tr}\left(\left(A^\top R^{-1} A \right)^{-1} \right) \\ \text{subject to} & R \in \mathcal{R}. \end{array} \tag{1}$$

Formulate (1) as a SDP.

Hint: learn about Schur complements.

Problem B. *(A simple SDP)* Consider the subset of $n \times m$ matrices

$$\mathcal{A} = \left\{ A \,:\, \|a_i - c_i\| \le R_i, \text{for } i = 1, \ldots, n \right\}.$$

Here, a_i denotes the ith column of A. The vectors $c_i \in \mathbb{R}^n$ and the scalars $R_i > 0$ are given. Let B be a given $n \times m$ matrix with full column-rank. Assume that $A^\top B + B^\top A$ is positive definite for all $A \in \mathcal{A}$. We want to solve

$$\begin{array}{ll} \text{minimize} & \mathrm{tr}\left(A \left(I_m + B^\top A + A^\top B \right)^{-1} A^\top \right) \\ \text{subject to} & A \in \mathcal{A}. \end{array} \tag{2}$$

The symbol I_m denotes the $m \times m$ identity matrix. Formulate (2) as a SDP.

References

[1] S. Boyd and L. Vandenberghe. *Convex optimization*. Cambridge University Press, 2004.