Nonlinear Optimization (18799 B, PP)
IST-CMU PhD course
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Instructor: `jxavier@isr.ist.utl.pt`
TAs: `augustos@andrew.cmu.edu`, `ric.s.cabral@gmail.com`

Important: The homework is due April, 2.

Homework 3

Instructions: read sections 4.1, 4.2, 4.3, and 4.4 of [1].

Problem A. *(Geometric approximation)* Let $P = \{x : Ax \leq b\}$ be a given polyhedron in $\mathbb{R}^n$. Assume that P is bounded and contains the origin. Let x_i, $i = 1, 2, \ldots, m$, be given points in $\mathbb{R}^n$. We are interested in sliding the polyhedron P in order to "cover" the given points as best as possible: we want to solve

$$\begin{array}{ll} \text{minimize} & \frac{1}{m}\sum_{i=1}^m d(x_i, c+P)^2 \\ \text{subject to} & c \in \mathbb{R}^n. \end{array} \tag{1}$$

Here, $c + P = \{c + x : x \in P\}$ is the polyhedron P translated by the vector c. Also, the notation $d(x, S)$ represents the distance from the point x to the set S,

$$d(x, S) = \inf \{\|x - s\| \,:\, s \in S\}\,.$$

In words, we want to translate P by a vector c such that the average squared distance from the given points to the shifted polyhedron is minimized. For example, if the given polyhedron were the singleton $P = \{0\}$ the optimal c would be $\frac{1}{m}\sum_{i=1}^m x_i$, the center of mass of the given points.

(a) Formulate (1) as a QP.

(b) Generate an instance of this problem for $n = 2$, i.e., set up appropriate A, b, and x_i, for $i = 1, 2, \ldots, m = 10$. Then, solve numerically the instance by using your QP formulation from part (a) and the software CVX from `http://cvxr.com/cvx`. Show a picture (e.g., a figure in Matlab) of your polyhedron P, the points x_i, and the optimal translate $c+P$.

Note: if you didn't solve part (a), then generate an instance of the QP (8.4), page 403 in [1], and solve it numerically through CVX. Plot both polyhedra and the corresponding closest points.

Problem B. *(Optimal control)* Consider two vehicles. The dynamics of vehicle i is given by

$$x_i(t+1) = A_i x_i(t) + b_i u_i(t), \quad t = 0, 1, 2, \ldots, T-1.$$

Here, $x_i(t) \in \mathbb{R}^n$ denotes the state of vehicle i, and $u_i(t)$ is its control signal. Both $A_i \in \mathbb{R}^{n\times n}$ and $b_i \in \mathbb{R}^n$, and the initial states $x_i(0)$ are given for $i = 1, 2$.

We want to design the control signals $u_i(t)$, $t = 0, 1, \ldots, T-1$, for $i = 1, 2$, in order to place vehicle i as close as possible to a given target position $p_i \in \mathbb{R}^n$ by time T. More precisely, we want to minimize $\|x_1(T) - p_1\|^2 + \|x_2(T) - p_2\|^2$.

There is a proximity constraint and a fuel constraint. At any time $t = 0, 1, 2, \ldots, T$, the distance between the two vehicles cannot strictly exceed a given threshold $R > 0$ (assume this holds for $t = 0$, i.e., $\|x_1(0) - x_2(0)\| \leq R$). Also, the fuel consumed by each control signal $u_i(t)$, $t = 0, 1, \ldots, T-1$, is given by

$$\sum_{t=0}^{T-1} \phi\left(u_i(t)\right)$$

and cannot strictly exceed a given budget B. Here, ϕ is a function mapping control amplitude to amount of fuel consumed, and is given by

$$\phi(u) = \begin{cases} |u|, & \text{if } |u| \leq 1 \\ u^2, & \text{if } |u| \geq 1 \end{cases}.$$

Formulate this optimal control problem as a LP, QP, or SOCP.

References

[1] S. Boyd and L. Vandenberghe. *Convex optimization.* Cambridge University Press, 2004.