Nonlinear Optimization (18799 B, PP)
IST-CMU PhD course
Spring 2013
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Important: The homework is due March, 15.

Homework 2

Instructions: read sections 3.1, 3.2, 3.3, and 3.5 of [1].

Problem A. *(Convex functions)*

(a) Let $f : \mathbb{R}^n \to \mathbb{R}$, $f(x) = x^\top Ax + b^\top x + c$, where $A \in \mathbb{R}^{n\times n}$, $b \in \mathbb{R}^n$, and $c \in \mathbb{R}$ are given. Moreover, the following property holds:

$$x^\top Ax \geq 0, \quad \text{for all } x \in \mathbb{R}^n.$$

Note that we are not assuming that A is a symmetric matrix. Show that f is a convex function.

(b) Let $C \subset \mathbb{R}^n$ be a closed convex set containing the origin as an interior point. The function $f : \mathbb{R}^n \to \mathbb{R}$ defined by

$$f(x) = \inf\{t > 0 \,:\, x \in tC\}$$

is called the *gauge* of C. Show that f is a convex function.

Hint: start by showing that f is positively homogeneous, i.e., $f(\alpha x) = \alpha f(x)$, for $\alpha \geq 0$ and $x \in \mathbb{R}^n$; then, prove that f is subadditive, i.e., $f(x+y) \leq f(x) + f(y)$, for $x, y \in \mathbb{R}^n$.

(c) Let $g : \mathbb{R}^n \to \mathbb{R}$ be a convex, continuous function. Show that the function $f : \mathbb{R}_{++} \to \mathbb{R}$

$$f(x) = \inf\{g(y) \,:\, \|y\| \leq x\},$$

is convex.

Hint: use Weierstrass' extreme value theorem; it states that a continuous function attains its infimum over any compact (closed, bounded) set.

A side note: the assumption on continuity of g is actually superfluous, since any *finite-valued* convex function on $\mathbb{R}^n$ is automatically continuous.

(d) Let $a \in \mathbb{R}^n$. Show that the function $f : \mathsf{S}^n_{++} \to \mathbb{R}$,

$$f(X) = a^\top X^{-1} a,$$

is convex.

(e) Let $A \in \mathsf{S}^n_{+}$. Show that the function $f : \mathsf{S}^n_{++} \to \mathbb{R}$,

$$f(X) = \exp\left(\operatorname{tr}\left(X^{-1}A\right)\right),$$

is convex.

Hint: use part (d) and the fact that the matrix A can be written as $A = BB^\top$ for some $B \in \mathbb{R}^{n\times n}$.

Problem B. *(Projections)*

(a) Let C and D be closed, convex subsets of $\mathbb{R}^n$ with non-empty intersection. Is it true that $p_{C\cap D}(x) = p_C\left(p_D(x)\right)$ for any x? In words, can we find the projection of a given $x \in \mathbb{R}^n$ onto the intersection $C \cap D$ by first projecting onto D, and then onto C? You should either prove the result, or find a counter-example.

(b) Consider the polyhedron $P = \left\{(x_1, x_2) \in \mathbb{R}^2 : x_1 \geq 0, x_2 \geq 0, x_1 + x_2 \leq 1\right\}$. The set $C = \left\{x \in \mathbb{R}^2 : p_P(x) = (0,1)\right\}$ denotes the set of points whose projection onto P is $(0,1)$. Make a sketch of both P and C.

(c) Consider the set of monotonically non-decreasing signals of length n, i.e.,

$$K = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 \leq x_2 \leq \cdots \leq x_n\}.$$

Note that K is a closed, convex cone. Let C denote the set of points whose projection onto K is the origin. Show that

$$C = \{(x_1, \ldots, x_n) \in \mathbb{R}^n : x_1 + \cdots + x_n = 0 \text{ and } x_i + \cdots + x_n \leq 0 \text{ for } i = 2, \ldots, n\}.$$

Problem C. *(Perron's theorem)*

(a) Let $x = (x_1, \ldots, x_n)$ be given, and let $m = \min\{x_1, \ldots, x_n\}$ and $M = \max\{x_1, \ldots, x_n\}$. Assume that $m < M$, that is, the vector x contains at least two distinct coordinates. Let $\lambda_i > 0$ for $i = 1, \ldots, n$, and $\lambda_1 + \cdots + \lambda_n = 1$. Show that

$$m < \sum_{i=1}^{n} \lambda_i x_i < M,$$

that is, any convex combination of the x_i's with *positive* weights cannot touch the extremes m and M.

(b) Let $P = (P_{ij})$ be an $n \times n$ positive matrix, i.e., $P_{ij} > 0$ for $1 \leq i, j \leq n$. Moreover, all rows of P sum to one,

$$P1 = 1. \tag{1}$$

Equation (1) shows that the vector 1 is a right-eigenvector of P, associated with the eigenvalue 1. It follows from linear algebra that P has a left-eigenvector, say $q \in \mathbb{R}^n$, associated with the eigenvalue 1,

$$q^\top P = q^\top. \tag{2}$$

Use Farkas' lemma to prove that, in fact, we can choose q to be positive; that is, show that there exists $q > 0$ such that (2) holds.

Hint: a vector v is positive if and only if there exists a positive scalar δ such that $v \geq \delta 1$.

A side note: the fact that q can be chosen to be positive is a consequence of Perron's theorem, a famous result in the theory of nonnegative matrices. Thus, we have asked you to prove (part of) Perron's theorem via Farkas' lemma.

References

[1] S. Boyd and L. Vandenberghe. *Convex optimization.* Cambridge University Press, 2004.