Problem set #6

Maximum Likelihood Method

1. Consider a sequence of iid (independent and identically distributed) observations $x_1, x_2, \ldots, x_N$. Determine the log-likelihood function and the ML estimate of the unknown parameters for each of the following cases

i) exponential distribution :	$p(x_i) = \frac{1}{\beta} e^{-\frac{x_i}{\beta}} u(x_i)$	parameter β
ii) Rayleigh distribution :	$p(x_i) = \frac{x_i}{f} e^{-\frac{x_i^2}{2f}} u(x_i)$	parameter f
iii) binomial distribution :	$p(x_i) = \binom{n}{x_i} \alpha^{x_i} (1-\alpha)^{n-x_i}$	parameter α
iv) normal distribution :	$p(x_i) = \frac{1}{\sqrt{2\pi\sigma^2}} e^{-\frac{(x_i-\mu)^2}{2\sigma^2}}$	parameters μ, σ^2

2. Check if the ML estimators derived in the previous problem are efficient i.e., if they achieve the Cramer-Rao lower bound.

3. Consider a data sequence generates by the model

$$x(n) = As(n) + w(n)$$

where $s(n)$ is known and $w(n)$ is a white Gaussian noise with distribution $w(n) \sim N(0, A)$. The parameter A controls both the mean and variance of the observations A. Find the ML estimate of A

4. Consider a sequence of dependent ramdom variables $x_1, x_2, \ldots, x_N$. Show that the joint pdf can always be decomposes as as a product of conditional pdfs as follows

$$p(x_1, \ldots, x_N) = p(x_1) \prod_{t=2}^{N} p(x_t | x_{t-1}, \ldots, x_1)$$

5. Consider a random signal $x_1, x_2, \ldots, x_N$ generated by an AR model

$$x(n) = ax(n-1) + w(n)$$

where $w(n) \sim N(0, \sigma^2)$ is a white Gaussian process and $x(1)$ is independent of a. We further assume that $w(n)$ and $x(1)$ are indenependent. Determine the ML estimate for a.

6. Repeat the previous problem assuming an AR model of order p

$$x(n) = a_1 x(n-1) + \cdots + a_p x(n-p) + w(n)$$

7. Consider a data set modeled by

$$x(n) = \phi(n)^T a + w(n) \qquad n = 1, \ldots, N$$

where $\phi(n) \in \mathbb{R}^p$ is a known vector, $w(n) \sim N(0, \sigma^2)$ is a white noise process and $a \in \mathbb{R}^p$ is an unknown vector to be estimated. Determine the ML estimate of a.

8. A radar system which radiates a known signal s(t) and receives an atenuated echo corrupted by noise. Let us assume that the received signal after sampling is

$$x(n) = As(n - \tau) + w(n) \qquad 1, \ldots, N$$

We will assume that $w(n) \sim N(0, \sigma^2)$ is a white Gaussian noise.

i) write the log-likelihood function for the estimate of A, τ

ii) can this function be analytically optimizes in general? under what conditions can you compute the optimal delay?

iii) assume that you know a close estimate of the delay τ_0 and derive a recursive algorithm for the optimization of the log-likelihood function.

9. (Listen to the baby) Consider an unborn baby inside her mother uterus. Suppose we record an acoustic signal of the mother's heart and also a mixture of moth heart signals, using two microphones. Therefore, we observe

$$\begin{array}{l} u(n) = x(n) + \eta(n) \\ v(n) = \alpha x(n) + \beta y(n) + \xi(n) \end{array} \qquad n = 1, \ldots, N$$

where $x(n), y(n)$ are the mother and baby heart signals and $\eta(n), \xi(n)$ are two white Gaussian noise processes with known distribution $N(0, \sigma^2)$.

i) estimate the unknown coefficients α, β using the ML method.

ii) devise a method to isolate the baby heart signal $y(n)$.

Bom trabalho!
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