Problem set #5

Probability Theory and Cramer-Rao Bound

Probability Theory

1. Consider a random variable x with normal distribution $N(\mu, \sigma)$. Determine the moments $E\{x\}, E\{(x-\mu)^2\}$.

2. Consider a random variable x with exponential distribution $p(x) = \frac{1}{\beta}e^{-x/\beta}u(x)$ with parameter β. Determine the moments $E\{x\}, E\{x^2\}$.

3. Consider a random variable x with Rayleigh distribution $p(x) = \frac{x}{f}e^{-\frac{x^2}{2f}}u(x)$ with parameter f. Determine the moments $E\{x\}, E\{x^2\}$.

Cramer-Rao Lower Bound

1. Consider a sequence of iid random variables $x_1, x_2, \ldots, x_n$ with exponential distribution $p(x) = \frac{1}{\beta}e^{-x/\beta}u(x)$ with parameter β.

   i) Determine the Cramer-Rao lower bound.

   ii) Consider the estimator $\hat{\beta} = \bar{x}$ where $\bar{x}$ is the arithmetic mean of the observations. Determine the variance of $\hat{\beta}$ and check if the estimator is efficient.

2. Repeat the previous problem assuming that the distribution is parametrized in terms of $\alpha = 1/\beta$ i.e., $p(x) = \alpha e^{-\alpha x}u(x)$. Consider the estimator $\hat{\alpha} = 1/\bar{x}$. Compute the Cramer-Rao bound and check if it is still possible to compute the variance of the estimator.

3. Consider a sequence of iid random variables $x_1, x_2, \ldots, x_n$ with Rayleigh distribution with parameter f.

   i) Determine the Cramer-Rao lower bound.

   ii) Consider the estimator

   $$\hat{f} = \frac{1}{2N}\sum_{k=1}^{N} x_k^2$$

   Determine its variance and check if it achieves the CRLB i.e., if the estimator is efficient.

4. Consider a sequence of iid random variables $x_1, x_2, \ldots, x_n$ with normal distribution $N(\mu, \sigma^2)$.

   Determine the Cramer-Rao lower bounds for the estimates of μ and σ^2.(we assume that we know one of the parameters and estimate the other)

   Consider the estimators $\hat{\mu} = \bar{x}, \hat{\sigma}^2 = \sum_{k=1}^{N}(x_k - \mu)^2$. Determine the variance of both estimators and check if they achieve the CRLB.

5. Consider a sequence of iid random variables $x_1, x_2, \ldots, x_n$ with normal distribution $N(\mu, \sigma^2)$.

   Determine the information matrix and Cramer-Rao lower bound for the joint estimation of μ and σ^2.

   Consider the estimators $\hat{\mu} = \bar{x}, \hat{\sigma}^2 = 1/N \sum_{k=1}^{N} (x_k - \mu)^2$. Determine the variance of both estimators and check if they achieve the CRLB.

6. Consider a sequence of iid discrete variables $x_1, x_2, \ldots, x_n$ and suppose that each variable may have M possible values with probability $P(x_k = i) = \alpha_i$. Determine Fisher information matrix and the Cramer-Rao lower bound for the estimation of the α's.

Bom trabalho!
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