Problem set #2

Discrete Fourier Transform

1. Identify the differences between the Fourier transform of a discrete signal and the discrete Fourier transform (DFT).

2. A finite length signal $x(n)$ (length $N = 4$) has DFT coefficients $X(0) = 1, X(1) = 1 + j, X(2) = 0, X(3) = 1 - j$. Determine $x(n)$.

3. Consider the following finite-length signals (length N) and determine their DFT.

$$\begin{array}{llll} a) & x(n) = 1R_N(n) & d) & x(n) = e^{j\frac{2\pi}{N}k_0 n}R_N(n) \\ b) & x(n) = \delta(n-m), \quad 0 \le m < N & e) & x(n) = e^{j\omega n}R_N(n), \quad \omega \neq \frac{2\pi}{N}k, \forall k \\ c) & x(n) = \delta(n) + 2\delta(n-1) & f) & x(n) = (-1)^n R_N(n) \end{array}$$

4. Determine a signal $x(n)$ with length $N = 4$, knowing that $x(1) = 1, x(3) = 0, X(1) = j, X(3) = -j$.

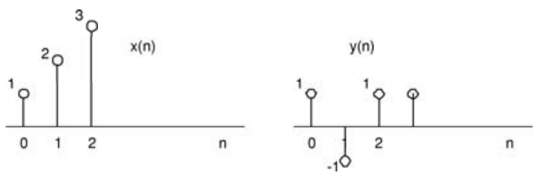
5. Consider the signal $x(n) = \delta(n-1) - \delta(n-3)$. Find an analytic expression for the periodic extension of $x(n)$ with period $N = 8$. Represent both signals graphically.

6. Consider a finite length signal $x(n)$ with DFT of length $N = 4$: $X(0) = 1, X(1) = j, X(2) = -1, X(3) = -j$. Determine its
   a) z transform
   b) DFT of length $N = 8$

7. Prove the following properties of the DFT. Assume that $x(n)$ is a discrete signal with finite length N and DFT $X(k), k = 0, \ldots, N-1$.

$$\begin{array}{lrcl} a) & x((n+m))_N R_N(n) & \leftrightarrow & e^{j\frac{2\pi}{N}km}X(k) \\ b) & e^{j\frac{2\pi}{N}k_0 n}x(n) & \leftrightarrow & X((k+k_0))_N R_N(k) \\ c) & x^*(n) & \leftrightarrow & X^*((-k))_N \\ d) & x^*((-n))_N & \leftrightarrow & X^*(k) \end{array}$$

8. Givan the following signals $x(n), y(n)$, determine the linear and circular convolution of length 5.



In which points the circular and linear convolutions are the same.

9. Repeat the previous exercise $N = 7$.

10. Compute the DFT of the previous signals and the DFT of their circular convolution $z(n)$. Show that $Z(k) = X(k)Y(k)$.

11. Given a signal $x(n)$ with length N, we wish to compute its DFT of length $2N$ using DFTs of length N. How can we do it?

12. A periodic signal $x(n)$ with period N is applied to the input of a LTI system with impulse response $h(n) = a^n u(n)$.

    a) prove that the filter output is also periodic with period N

    b) show that there exists a LTI filter with finite impulse response of length N which produces the same output as $h(n)$. Determine this filter.

    c) compute the frequency response of both filters at the frequencies $\omega = \frac{2\pi}{N}k$. What do you conclude?

(extension problems - vector spaces of signals)

13. Prove that the set S of discrete signals with suport $\{0, \ldots, N-1\}$ is a vector space of dimension N.

14. Consider the following sets of finite length signals

    impulse signals: $\phi_k(n) = \delta(n-k), \quad k = 0, \ldots, N-1$

    harmonic signals: $\phi_k(n) = \frac{1}{\sqrt{N}} e^{j\frac{2\pi}{N}k} R_N(n), \quad k = 0, \ldots, N-1$

    Show that each of these sets is a base for the space of signals with finite length N.

    Hint: prove that they are linearly independent.

15. Consider the following inner product defined in the space of signals with finite length N

$$< x, y >= \sum_{n=0}^{N-1} x(n) y(n)^*$$

    Prove that the two sets of signals (impulses, harmonics) defined above are orthonormal i.e., $< \phi_i, \phi_j >= \delta(i-j)$

16. Prove that if S is a vector space of finite dimension N and $B = \{\phi_i, i = 0, \ldots, N-1\}$ is an orthonormal base of S then each vector (signal) $x \in S$ can be represented as follows

$$x(n) = \sum_{k=0}^{N-1} c_i \phi_i(n) \qquad c_i =< x, \phi_i >$$

17. Prove that the DFT computes the set of c_i coeffiencients (apart from a scale factor) for the harmonic base functions.

18. What are the coefficients if we adopt the impulse base functions. Do you think this is an interesting transform?

Bom trabalho!
Jorge Salvador Marques