The Fokker-Planck equation in estimation and control [⋆]

João M. Lemos [*] **João Girão** [**] **António S. Silva** [***]
Jorge S. Marques [****]

[*] *INESC-ID, IST, Univ. Lisboa, Portugal (e-mail: jlml@inesc-id.pt).*
[**] *INESC-ID, IST, Univ. Lisboa, (e-mail: joao.girao@ist.utl.pt)*
[***] *INESC-ID, IST, Univ. Lisboa, (e-mail: antssilva96@gmail.com)*
[****] *ISR, IST, Univ. Lisboa (e-mail: jsm@isr.tecnico.ulisboa.pt*

Abstract: In this work it is shown how the Fokker-Planck equation can be used to address the solution of problems in a variety of fields in which a set of dynamical agents is concerned, including continuous discrete filters, adaptive control, and tracking groups of targets.

Keywords: Nonlinear estimation, Fokker-Planck equation, stochastic agents, group estimation, dual adaptive control.

1. INTRODUCTION

The Fokker-Planck Equation (FPE) provides a link between a continuous Markov stochastic system, described by a stochastic differential equation and the time evolution of the probability density function (pdf) of its state (Jazwinski (1970)). Applications cover a diversity of areas, such as telecommunications (Viterbi (1963)), agriculture (Scheerlinck *et al.* (2004)), molecular biology (Smith (2002)), or fire propagation (Séro-Guillaume *et al.* (2008)). From another point of view, the FPE may be considered as a way of describing the expected dynamic behaviour of an ensemble of stochastic agents. An example is provided by a large number of mobile robots that are to be controlled such as to achieve a specified spatial distribution. This example, thoroughly treated in Foderaro *et al.* (2014) may be transposed to other application areas, such as internet congestion (Mukherjee and Strikwerda (1991)), or avoiding overloads in a power appliance population.

The early references Jazwinski (1966, 1970) propagate the state pdf using the FPE and then use Bayes law to filter this predicted density using the observations,. More recently, this subject receives significant attention, including the use of Feynman path integrals to approximate solutions to the FPE (Balaji *et al.* (2008); Balaji (2009)), kernel density estimates again together with path integrals (Singer (2003)), and fast approximations (Mazzoni (2012)).

The techniques addressed in this article may be seen as a means to study networks of interacting agents whose collective dynamics emerge from a large number of ensemple members. The present work addresses problems in which the FPE propagates in time a pdf that reflects the *a priori* expected behaviour of a population of agents from which the evolution of a single agent is then individuated through observations made with appropriate sensors. For this purpose, a way to obtain the approximate

solution of the FPE in discrete time is presented. Two case studies are addressed. One consists of parameter estimation for the model of a single individual subject to general anaesthesia, drawn from a population for which a statistical characterization is available, and its extension to dual adaptive control. The second case study consists of the tracking of a target (*e. g.*, a person) that moves in a region in which multiple preferred paths are *a priori* known.

The paper is addressed as follows. After this introduction, the characterization of the motion state of stochastic agents in continuous and discrete time is presented in section 2, together with a discrete time approximation of the solution of the FPE. Section 3 presents the case study on target tracking, section 4 the one on parameter estimation and dual control. Finally, section 5 draws conclusions.

2. STOCHASTIC AGENTS AND THE FPE

This work considers stochastic agents described by the stochastic differential equation (SDE)

$$dx_t = f(x_t)dt + \sigma dw_t \qquad (1)$$

where σ is a constant parameter, $x \in \mathbb{R}^n$, the initial condition $x(0) = x_0$ is a random variable with pdf p_{x_0}, and w_t is a Wiener process such that $\mathcal{E}(dw_t dw_t^T) = Qdt$, with Q a constant matrix. Under this assumption, (1) is the same in either the Itô or Stratonovitch sense. For $t > 0$ the pdf $p(x,t)$ of the state x of the diffusion process satisfies the Fokker-Planck equation (FPE) Jazwinski (1970), given by

$$\frac{\partial p}{\partial t} = -\sum_{i=1}^{n} \frac{\partial}{\partial x_i}(pf_i) + \frac{\sigma^2}{2} \sum_{i,j=1}^{m} \frac{\partial^2}{\partial x_i \partial x_j}(pQ_{ij}). \qquad (2)$$

For x a scalar ($n = 1$), the FPE equation reduces to

$$\frac{\partial p}{\partial t} = -f_x(x)p - f(x)\frac{\partial p}{\partial x} + \frac{\sigma^2}{2}\frac{\partial^2 p}{\partial x^2}, \qquad (3)$$

with the initial condition

$$p(x, 0) = p_{x_0}(x), \qquad (4)$$

and the boundary conditions

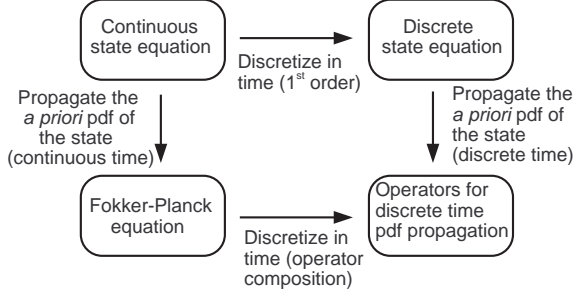$$p(\pm\infty, t) = 0, \quad \forall t > 0. \qquad (5)$$

Fig. 1. Probabilistic interpretation of discretizing the FPE

In order to obtain an approximate formula for the solution of the FPE over a small interval of time Δ, consider the operators T_Δ^i, $i = 1, 2, 3$, defined by

$$T_\Delta^1 p(x, t) \approx \frac{1}{1 + f_x(x)\Delta} p(x, t), \qquad (6)$$

$$T_\Delta^2 p(x, t) \approx p(x - f(x)\Delta, t), \qquad (7)$$

and

$$T_\Delta^3 p(x, t) = p(x, t) * G(x, \Delta), \qquad (8)$$

where $*$ stands for convolution and G is the Gaussian kernel given by

$$G(x, \Delta) = \frac{1}{(2\pi\sigma^2\Delta)^{1/2}} \exp\left(-\frac{x^2}{2\sigma^2\Delta}\right). \qquad (9)$$

Then, up to first order terms in the time increment Δ, the following approximate solution of the FPE holds,

$$p(x, t + \Delta) \approx T_\Delta^3 T_\Delta^2 T_\Delta^1 p(x, t). \qquad (10)$$

The approximation (10) is justified by the so-called "Trotter's formula" (Trotter (1959)), being valid under conditions trhat are verified by the operators T^1, T^2 and t^3. Although the convergence of this approximation is only linear in the time step Δ, it has the advantage of allowing the probabilistic interpretation shown in figure 1, in addition to its simplicity.

On the other way, sample now the SDE 1 using the first order Euler-Maruyama method Higham (2001) to obtain the stochastic difference equation:

$$x_{k+1} = x_k + f(x_k)\Delta + \sigma(w_{k+1} - w_k) \qquad (11)$$

where $x_k := x(k\Delta)$, $w_k := w(k\Delta)$ and $\Delta \in \mathbb{R}$ is the time discretization step. The solution of (11) converges to the solution of (1) in mean square when $\Delta \to 0$ Jazwinski (1970).

Furthermore, for Δ small, the operators that propagate in discrete time the pdf of the state of the discrete model (11) are the same as the ones that approximate the solution of the FPE in (10). Figure 1 illustrates this fact by a block diagram: starting with a SDE that defines the state evolution in continuous time, one may either sample it to obtain an approximate discrete state equation and then propagate in time the state pdf, or propagate the state in continuous time, using the FPE and, finally, approximate the solution of the FPE by applying convenient operators, that yield the same result.

An example is provided by the PLL error dynamics Viterbi (1963), where the vector field of the SDE is given by

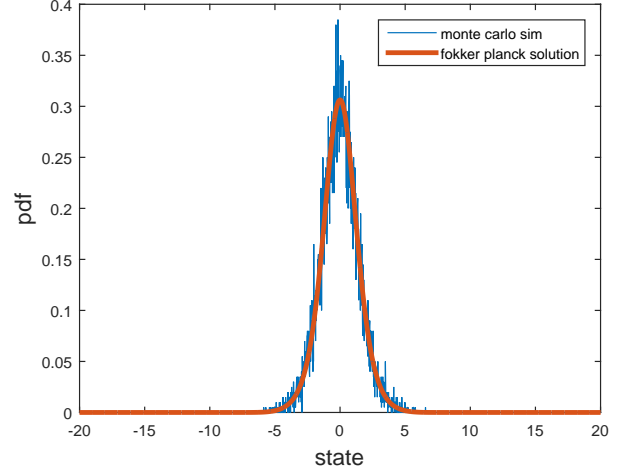$$f(x) = ax - K_{PLL}sin(x), \qquad (12)$$

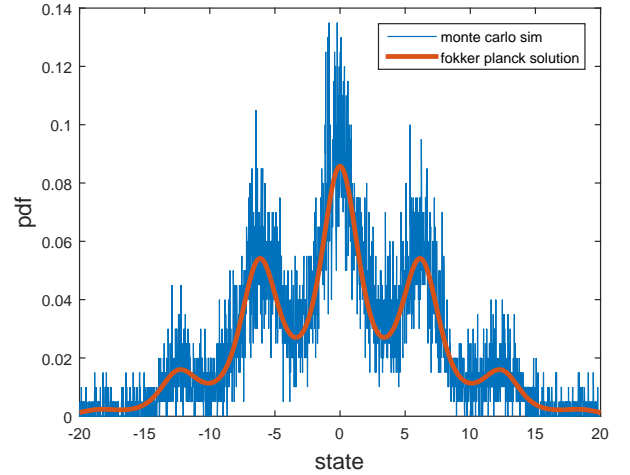Fig. 2. Solution of the FPE for the PLL and relative frequency plot of Monte Carlo simulations for $t = 20$

Fig. 3. Solution of the FPE for the PLL and relative frequency plot of Monte Carlo simulations for $t = 700$

and $a \leq 0$ and K_{PLL} are constant parameters. Figures 2 and 3 show the solution of the FPE given by (10), at two different time instants, superimposed on the relative frequency plot of the values of the state of the difference equation, obtained in a Monte Carlo simulation with 1000 runs. These results are shown just to illustrate the two ways of propagating the state of a stochastic dynamical system. In a practical situation, it is advantageous to use the FPE or the operators that propagate its solution to obtain the state pdf, rather then resorting to Monte Carlo simulations with (11). Furthermore, although there are numerical methods to solve the FPE, such as the Cranck-Nicholson method, see e. g. Keller (1960), in which the error convergence is of second order, the method proposed has the advantage of both being explicit as well as providing a direct link with the dynamic state equations (a pdf shift, associated to the vector field f, that corresponds to operators T^1 and T^2, and a convolution with a Gaussian kernel, as defined by T^3, that corresponds to the stochastic term).

3. TARGET TRACKING

Target (including pedestrians or surface vehicles, such as cars or bicycles) surveillance in far field settings, *i. e.*, when the camera covers a wide field and it is impossible to extract detailed shape information about the object being supervised, is a subject that is currently attracting a significant attention Nascimento *et al.* (2013). This example is concerned with the surveillance of areas where the trajectories most frequently followed present a defined pattern. If the trajectories are generated by a stochastic motion dynamic model, as (1), the *a priori* pdf of target distribution as a function of time satisfies a FPE that is derived from the it.

Add now to (1) the observations model

$$y(t) = h(x(t)) + \eta(t), \tag{13}$$

where $y \in \mathbb{R}$ is the observation, $h : \mathbb{R}^n \to \mathbb{R}$ and $\{\eta\}$ is a sequence of independent, identically distributed Gaussian random variables with zero mean and variance σ_η^2. Assume that the observations are made at discrete time instants $t_0, t_1, \ldots, t_k$ and define the set of observations

$$Y^{t_k} = \{y(t_0), y(t_1), \ldots, y(t_k)\}. \tag{14}$$

With an abuse of notation we consider pdf conditioned on sets like Y^{t_k} where the conditioned should be on the σ-algebras associated to these sets. We consider the problem that consists of estimating $x(t_i)$ from the observations of y up to time t. Whatever the estimation criterion might be, the full information required to compute the estimate is contained in the pdf of x given the observations. This pdf is computed according to the following steps:

FPE based filter

Let $p(x(t_{k-1})|Y^{t_{k-1}})$ (the "filtered pdf") be available from the previous steps of the algorithm.

- **Prediction step:** Compute $p(x(t_k)|Y^{t_{k-1}})$ (the "predicted pdf") by propagating from time t_{k-1} until t_k the pdf $p(x(t_{k-1})|Y^{t_{k-1}})$. For this sake solve the FPE (3) in the time interval that starts at t_{k-1} and ends at t_k, taking as initial condition $p(x(t_{k-1})|Y^{t_{k-1}})$.
- **Filtering step:** Compute the filtered pdf at time t_k using

$$p(x(t_k)|Y^{t_k}) = \mathcal{K}(t_k)p(y(t_k)|x(t_k))p(x(t_k)|Y^{t_{k-1}}), \tag{15}$$

where $\mathcal{K}$ is a normalizing constant that depends on time.

□

The proof of (15) uses a well known argument based on Bayes law. The pdf $p(y(t_k)|x(t_k))$ depends on the observations (sensor) model. In the situation described by (13),

$$p(y(t_k)|z(t_k)) = C_\eta exp\left\{-\frac{1}{2\sigma_\eta^2}[y(t_k) - h(z(t_k))]^2\right\}, \tag{16}$$

where C_η is a normalizing constant.

For the sake of illustration, figure 4 shows an example with multiple target tracking, in just one dimension. The solution of the FPE is shown in the plane defined by time and the state x. The initial condition is a mixture of Gaussian functions that has two modes centered at $x = 5$ and $x = 7$ that correspond to the most probable regions from which targets start. Each of the pdf plots marked with blue lines corresponds to one iteration of (10). Every 5 discrete steps this *a priori* pdf is updated from the observation of three targets, the exact position of which
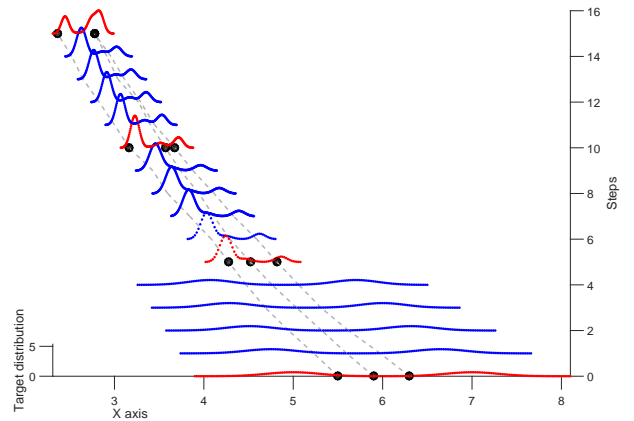

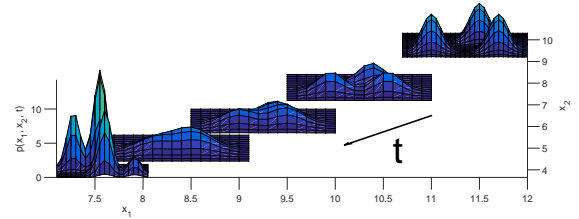Fig. 4. Target tracking in one dimension

Fig. 5. Target tracking in two dimensions

(marked with round bullets) is corrupted by Gaussian noise. A filtering step is then performed using (15), from which the red line pdf plots result.

Figure 5 shows the first 5 steps for a two dimensional state, as is the case when considering the position in a 2D scene.

4. PARAMETER ESTIMATION AND DUAL ADAPTIVE CONTROL

The previous ideas are now extended to the simultaneous estimation of parameters and state. Although more general situations can be considered, it is assumed that the parameters are constant, but unknown, and that the process noise can be neglected, the only source of uncertainty stemming from the initial condition that is assumed to have a known pdf. Consider, thus, a nonlinear system modeled by the state equation

$$\frac{dx}{dt} = f(x, \theta), \tag{17}$$

with state $x \in \mathbb{R}^n$, parameters $\theta \in \mathbb{R}^{n_p}$, $f : \mathbb{R}^n \times \mathbb{R}^{n_p} \to \mathbb{R}^n$.

The initial condition $x(t_0) = x_0$ is, in general, a random variable with pdf $p_x(x(t_0), t_0)$. Consider first the situation in which the parameter vector is constant, being a random variable for which the only *a priori* information about it is the pdf $p_\theta(\theta)$. Depending on the value of θ, different trajectories of the state x are obtained. At each time t, the state $x = x(t, \theta)$ is therefore also a random variable characterized by a pdf $p_x(x, t)$, even if $x(t_0)$ is deterministic. In order to compute $p_x(x, t)$, define the augmented state $z \in \mathbb{R}^{n+n_p}$ given by

$$z(t) = \begin{bmatrix} x(t) \\ \theta \end{bmatrix}. \tag{18}$$

The augmented state verifies the equation

$$\frac{dz}{dt} = F(z), \tag{19}$$

with

$$F(z) = \begin{bmatrix} f(x) \\ 0 \end{bmatrix},$$

and initial conditions that derive from (1) in a straightforward way. Computing $p(z,t)$ provides not only the required information on $p_x(x,t)$ but also on $p_\theta(\theta,t)$.

Under the modelling assumptions made, the FPE reduces to the so-called Liouville equation

$$\frac{\partial}{\partial t} p(z,t) + \sum_{i=1}^{n+n_p} \frac{\partial}{\partial z_i} (F_i(z)p(z,t)) = 0, \tag{20}$$

or, in a more compact way,

$$\frac{\partial p}{\partial t} + \frac{\partial p}{\partial z} F + tr\left(\frac{\partial F}{\partial z}\right) p = 0, \tag{21}$$

where

$$\frac{\partial p}{\partial z} = \begin{bmatrix} \frac{\partial p}{\partial z_1} & \cdots & \frac{\partial p}{\partial z_{n+n_p}} \end{bmatrix}$$

is the gradient of p with respect to z, and

$$\frac{\partial F}{\partial z} = \begin{bmatrix} \frac{\partial F_i}{\partial z_j} \end{bmatrix}$$

is the Jacobian matrix of F with respect to z at time t. The solution of (20) (or, equivalently, (21)) is subject to the initial condition $p(z,t_0)$ specified and to the boundary conditions $\lim z \to \pm\infty = 0$.

While the FPE is a parabolic equation, the Liouville equation is a hyperbolic equation. Moreover, the Liouville equation can be solved exactly using Laplace's method, Ibragimov (1999).

In order to illustrate this method, consider the neuromuscular blockade of patients subject to general anesthesia induced by *atracurium* administration, Lemos*et al.* (2014). The dynamic system for the neuromuscular blockade may be modeled by (22) and (23). Here, the parametrization proposed in Silva *et al.* (2012) is used. The linear part of the model, (22), relates the input of the system, *i.e.* the drug infusion rate or *atracurium* dosage $u(t)$ to the state variable $x_3(t)$.

$$\frac{dx}{dt} = \begin{bmatrix} -10\alpha & 0 & 0 \\ 4\alpha & -4\alpha & 0 \\ 0 & \alpha & -\alpha \end{bmatrix} x(t) + \begin{bmatrix} 10\alpha \\ 0 \\ 0 \end{bmatrix} u(t), \tag{22}$$

where $x(t) = [x_1(t), \ x_2(t), \ x_3(t)]$ and α is an unknown patient dependent parameter Silva *et al.* (2012).

The observation model is given by the Hill equation

$$y(t) = \frac{100}{1 + (x_3(t)/3.2425)^\gamma} + \eta(t), \tag{23}$$

where y is the observation of the blockade level r, γ is a patient dependent parameter Silva *et al.* (2012) and η is assumed as a Gaussian noise with standard deviation $\sigma_\eta = 3$.

Due to clinical reasons, for the population considered, the initial conditions are set as $x_1(0) = 5000\alpha$ and $x_2(0) = x_3(0) = 0$. The parameters α and γ : are assumed to have, respectively, the pdf $\log(\alpha) \sim N(-3.287, 0.158^2)$, $\log(\gamma) \sim N(0.9812, 0.3458^2)$.

The joint parameter and state estimation algorithm can be used to develop a dual adaptive controller for sample data
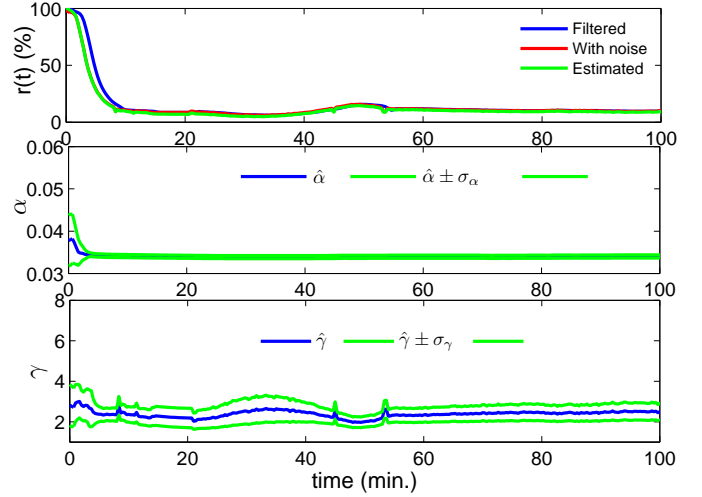

Fig. 6. Parameter estimation using the FPE based filter in a clinical data record of neuromuscular blockade induced by *atracurium*

nonlinear systems with unknown parameters. For that sake, consider the plant with unknown parameters modelled by (17) or, equivalently, (19), where z is a hyperstate Åstrom *et al.* (1999), but assume now that the vector field f also depends on a control variable u. Furthermore, assume that the plant is sampled with a sampling interval Δ. Let the performance index for control optimization be

$$J = E\left\{\sum_{k=0}^{N-1} L_{k+1}(x(k+1), u(k))\right\}, \tag{24}$$

with $L_{k+1}(\cdot, \cdot)$ known positive, convex, scalar functions. The problem of optimal dual control Åstrom *et al.* (1999) consists of finding the sequence of control variables $u(0)$, $u(\Delta)$, ..., $u((N-1)\Delta)$ that minimizes J and belong to the set of admissible controls. As is well known, the solution to this problem is given by the dynamic programming equation Filatov *et al.* (2004)

$$V_k(Y^k) = \min_{u(k)} E\left[L_k(x(k+1), u(k)) + V_{k+1}(Y^{k+1}|Y^k)\right], \tag{25}$$

with terminal condition

$$V_{N-1}(Y^{N-1}) = \min_{u(N-1)} E\left[L_N(x(N), u(N\,1))|Y^{N\,1}\right], \tag{26}$$

for $k = N-2, N-3, \ldots, 0$.

Although the FPE could ideally be used to compute the expectations in (25), (26), the "curse of dimensionality" prevents in general the numerical solution of these equations and leads to the consideration of approximate methods Filatov *et al.* (2004). Among these approximations, an effective one consists in combining a bicriterial approach with model predictive control Silva *et al.* (2005). Accordingly, the selection of the control variable is made by considering at each sampling interval $k\Delta$ the two cost functions

- J_k^c, the cautious control cost,
- J_k^a, the uncertainty cost,

according to the following steps

(1) Obtain the cautious control value $u_c(k)$ by solving

$$u_c(k) = \arg\min_{u(k)} J_k^c.$$

(2) Minimize J_k^a in the interval Ω_k given by
$$\Omega_k = [u_c(k) - \vartheta(k), \, u_c(k) + \vartheta(k)].$$

This interval is centered in the cautious control value $u_c(k)$, and its with is proportional to a measure of uncertainty of the parameters computed friom the solution of the FPE. Therefore, the approximate dual control law actually applied to the plant at the sampling interval $k\Delta$ is
$$u(k) = \arg \min_{u(k) \in \Omega_k} J_k^a.$$

Different possibilities may be considered to J_k^c and J_k^a. One of them, that inherits the good qualities of model predictive control is to make
$$J_c^k = E\left\{ \sum_{i=1}^{N} (y(k+i) - r(k+i)^2 + \rho u^2(k+i-1)|Y^k \right\},$$
with $\rho \geq 0$ and tr tyhe reference to track, and
$$J_k^a = -E\left\{ \sum_{i=1}^{N} (y(k+i) - \hat{y}(k+i))^2 + \right.$$
$$\left. \alpha(u(k+i-1) - \hat{u}(k+i-1))^2 |Y^k \right\},$$
with $\alpha \geq 0$ and $\hat{y}$ and $\hat{u}$ the estimates of y and u.

In Silva *et al.* (2005), this algorithm has been developed by considering linear models and an index of parameter uncertainty yielded by the recursive least squares estimation algorithm. Here, instead, the extension to the nonlinear case is made possible through the use of the FPE.

5. CONCLUSIONS

The Fokker-Planck equation provides a mean to describe ensembles of stochastic agents that can be applied to a variety of problems, that range from propagating the *a priori* pdf of the process state, propagate the filtered state *a posteriori* pdf (*i. e.*, given the observations), joint estimation of process state and parameters, and adaptive control. Case studies concerned with target tracking and model identification for patients subject to anaesthesia are described to illustrate these actions. The state estimation procedure based on the FPE is tightly related to particle filtering (PF). The main difference is that, using the FPE, there is no need to perform computationally heavy Monte Carlo simulations, a task that is replaced with the solution of the FPE, for which an approximate solution in discrete time is presented. When considering adaptive control problems, the FPE based approach to estimate the state and parameters has the advantage of providing a characterization of the uncertainty of the estimates that can be used to develop suboptimal dual adaptive controllers.

REFERENCES

Åstrom, K. J. and B. Wittenmark, *Adaptive Control*, Addison-Wesley, 1989.

B. Balaji and M. McDonald (2008). Continuous-Discrete Filtering Using the Dirac-Feynman Algorithm. *Proc. IEEE Radar Conf., RADAR 2008*, pp. 1-6.

B Balaji (2009). Continuous-Discrete Path Integral Filtering. *Entropy*, 11:402-430, 2009.

Filatov, N. M. and H. Hunbehauen (2004), *Adaptive Dual Control*, Springer, 2004.

Foderaro, G.; S. Ferrari and T. A. Wettergren (2014). Distributed optimal control for multi-agent trajectory optimization. *Automatica*, 50: 149-154, 2014.

Higham, D.J. (2001), An Algorithmic Introduction to Numerical Solution of Stochastic Differential Equations, *SIAM REv.* 43(3): 525-546.

Ibragimov, N. H. (1999), *Elementary Lie Group Analysis and Ordinary Differential Equations*, Wiley, 1999.

Jazwinski, A. H. (1966) Filtering for Nonlinear Dynamical Systems. *IEEE Trans. Autom. Control* 11: 765-766, 1966.

Jazwinski, A. H. (1970), *Stochastic Processes and Filtering Theory, Academic Press*, 1970.

Keller, H. B. (1960). The numerical solution of parabolic partial differentiaçl equations, chapter 12 of A. Ralston and H. S. Wiff (eds.), *Mathematical methods dor digital computers*, John Wiley & Sons, 1960.

Lemos, J. M.; D. V. Caiado, B. A. Costa, L. A. Paz, T. F. Mendonça, R. Rabiço, S. Esteves and M. Seabra (2014). Robust Control of Maintenance Phase Anesthesia. *IEEE Control Systems Magazine*, 34(6):24-38, 2014.

Mazzoni, T. (2012). Fast continuous-discrete DAF-filters. *J. Time Ser. Anal.* 33: 193-210, 2012.

Mukherjee, A. and J. C. Strikwerda (1991). Abnalysis of dynamic congestion control protocols – A Fokker-Planck approximation. *Proc. ACM Sigcomm*, 159-169, Zurich, Switzerland.

Nascimento, J. C., J. S. Marques and J. M. Lemos (2013). Modelling and classifying human activities from trajectories using a class of space-varying parametric motion fields. *IEEE Trans. Image Processing*, 22(5):2066-2080.

Scheerlinck, N.; A. Peirs, M. Desmet, A. Schenk and B. M. Nicolai (2004), Modelling Fruit Characteristics During Apple Maturation: A Stochastic Approach. *Mathematical and Computer Modelling of Dynamical Systems*, 10(2):149-168.

Séro-Guillaume, O.; S. Ramezani, J. Margerit and D. Calogine (2008), On large scale forest fires propagation models. *Int. J. of Thermal Sciences*, 47:680-694.

Silva, M. M.; T. Wigren and T. Mendonça (2012). Nonlinear Identification of a Minimal Neuromuscular Blockade Model in Anesthesia. *IEEE Transactions on Control Systems Technology*, 20(1):181-188, 2012.

Silva, R. N.; N. Filatov, J. M. Lemos and H. Unbehauen (2005). A dual approach to start-up of an adaptive predictive controller. *IEEE Trans. Contrpol Systems Techon.*, 13(6): 877-883.

Singer, H. (2003) Nonlinear continuous-discrete filtering using kernel density estimates and functional densities. *J. of Mathematical Sociology*, vol. 27, pp. 1-28, 2003.

Smith, G. D. (2002), Modeling the stochastic gating of ion channels. Chap. 11 of C. Fall, E. Marland, J. Wagner and J. Tyson (eds.), *Computational Biology*, Springer, New York.

Trotter, H. F. (1959) On the product of semigroups of operators. Proc. Am. Math. Soc., 545-551.

Viterbi, A. J. (1963) Phase-locked loop dynamics in the presence of noise by Fokker-Planck techniques. *Proc. IEEE*, 51:1737-1753.