Effective ADC Linearity Testing Using Sinewaves
Francisco André Corrêa Alegria, Antonio Moschitta, Paolo Carbone, António Manuel da Cruz Serra, and Dario Petri

Abstract—This paper deals with the effectiveness of the sinewave histogram test (SHT) for testing analog-to-digital converters. The implementation is discussed, with respect to the adopted procedures and to the choice of relevant parameters. Some of the published approximations currently limiting the characterization of the test performance are removed. The statistical efficiency of the SHT is evaluated by comparing the associated estimator variance with the corresponding Cramér–Rao lower bound, theoretically derived assuming sinewaves corrupted by Gaussian noise. Finally, both simulation and experimental results are presented to validate the proposed approach.

Index Terms—Cramér–Rao lower bound (CRLB), sinewave histogram test (SHT).

I. INTRODUCTION

THE widespread usage of analog-to-digital converters (ADCs) confers great importance to the testing activities, which nowadays largely contribute to the production costs of integrated circuits. Since the ADC test duration and costs grow significantly for high resolution ADCs, choosing an efficient test and improving the associated performance may significantly reduce the industrial cost of an ADC manufacturing process [1]. ADCs are usually characterized by figures of merit like effective resolution, signal-to-noise-and-distortion ratio (SINAD), integral nonlinearity (INL), or differential nonlinearity (DNL) [2]. In particular, INL and DNL are related to the accuracy of the converter transition voltages and code bin widths. So they are of great interest when the quality of the manufacturing process has to be controlled. Various ADC testing procedures have been proposed in the literature in order to estimate these parameters [1]. A popular method is the sinewave histogram test (SHT), which estimates INL and DNL related to each transition voltage of an ADC stimulated by a pure sinewave [3]–[6]. Accordingly, it is important to characterize properly the statistical properties of the INL and DNL estimators in order to establish both test performance parameters and corresponding upper bounds.

In this paper, the performance of the SHT is analyzed and discussed, taking into consideration both theoretical and practical aspects. Bias and variance of INL and DNL estimators are considered and used both to provide an effective ADC testing procedure and to compute the statistical efficiency of the transition level estimator. In particular, a practically relevant stimulus, consisting in an ideal sinewave corrupted by an additive white Gaussian noise (AWGN) is considered, and the effects of noise are analyzed. First, the SHT is introduced and described, and the main sources of uncertainty are identified. In particular, the statistical efficiency of the SHT is evaluated by comparing the variance of the achieved estimator with the corresponding Cramér–Rao lower bound (CRLB) [7]–[9] which describes the minimal achievable variance for any unbiased estimator. To this aim, the CRLB associated to the estimation of the transition levels of an ADC fed by a noisy stimulus has been theoretically modeled, both for biased and unbiased estimators. Then, the SHT implementation issues are discussed and improvements are presented that remove some of the published approximations currently limiting the characterization of the test performance. Finally, both simulation and experimental results are provided to validate the proposed results.

II. SHT ANALYSIS

A. Basics of SHT

The SHT is used to estimate the input–output characteristic of ADCs, that is the transition voltages and code bin widths, whose accuracy is generally expressed using INL and DNL, together with gain and offset errors. These parameters are estimated by comparing the number of samples counted in each ADC output code bin (histogram) when a sinusoidal input signal is employed, with that expected when assuming an ideal quantizer. In particular, the $k$th transition voltage $T_k$ is estimated by determining the probability of the input voltage being in the range $]-\infty, T_k[$, by means of the cumulative histograms $H_k$, that is, the number of acquired samples exhibiting output code equal to or lower than $k$. The expression for the transition voltages estimator $\hat{T}_k$ is [3], [6]

$$\hat{T}_k = C - A \cos \left(\frac{\pi H_k}{M}\right), \quad k = 1, \ldots, 2^N - 1$$

where $M$ is the total number of acquired samples, and $N$ is the ADC resolution (number of bits), while $C$ and $A$ are the input signal offset and amplitude, respectively, suitably chosen to stimulate all possible output codes.

1057-7122/$20.00 © 2005 IEEE
B. Accuracy and Statistical Efficiency of the Transition Level Estimator

The testing process, which seems to be easily implemented in practice, is subject to several accuracy limiting factors [2], [3]. The most important are the stimulus signal nonidealties, the uncertainty in the ratio between input sinewave frequency and ADC sampling rate and the finite number of acquired samples.

In fact, the stimulus signal usually exhibits distortions (mainly harmonics) which adversely affects the input voltage amplitude distribution. To minimize the consequences of this effect, a function generator should be used with a spurious-free dynamic range (SFDR) high enough to make the distortion error power negligible compared to the quantization noise power. This can be achieved by considering that an ideal N-bit quantizer, fed with a sinusoidal signal, exhibits a signal to quantization noise ratio of approximately $(6,02N + 1,76)$ dB.

An additional cause of deviation of the input signal from a pure sinewave is the wide-band noise introduced by test bed components and by input-referred ADC noise sources. This noise can be usually modeled as AWGN and introduces both a systematic and a random deviation on measurement results. Unfortunately, a closed-form expression for a transition level estimator that compensates for the systematic deviation, called bias, has not yet been derived. Thus, bias is usually reduced by overdriving the ADC, that is by applying a sinewave with an amplitude slightly exceeding the minimum one needed to stimulate all ADC codes. Overdriving the ADC is effective because, due to the sinewave amplitude distribution, systematic noise effects are more pronounced near the ADC full scale [4].

The accuracy of the transition voltage estimates may be also affected by uncertainties in the values of both input sinewave amplitude and offset, as can be directly deduced from (1). However, it has been shown that such contributions are usually negligible with respect to those introduced by phase and frequency uncertainties [4].

Furthermore, in order for the phase of the collected samples to be uniformly distributed, the sampling rate $f_s$ and the stimulus signal frequency $f$ should satisfy the following [3]:

$$\frac{f}{f_s} = \frac{J}{M}$$  \hspace{1cm} (2)

in which the integer number $J$ of acquired sinewave periods is assumed to be mutually prime with $M$ [2]. The requirement (2) can be satisfied using frequency-locked generators for both the input signal and the clock signal. In many practical applications, however, such constraint can not be met. If this is the case, the phases of the collected samples are not uniformly distributed, and an additional error is introduced in the estimation of the transition voltages [2]. Finally, the finite number of acquired samples also contributes to the estimator uncertainty, because of the random nature of both the unknown sinewave phase and the input additive noise.

In fact, provided that (2) is satisfied and that the bias of the $k$th transition level estimator is reduced to a negligible value by overdriving the ADC [5], the related variance $\sigma_{\hat{T}_k}^2$ can be expressed by [5]

$$\sigma_{\hat{T}_k}^2 \simeq \frac{1}{M^2} \left[ A^2 - T_k^2 \right] + \frac{1,78}{M} \sqrt{A^2 - T_k^2}$$

$$A \geq |T_k|,$$

$$k = 1, \ldots, 2^N - 1$$

$$\alpha_k \simeq \frac{2 \nu_k}{M}, \quad \nu_k \simeq \arccos \left( \frac{\bar{T}_k}{A} \right)$$

$$\bar{T}_k \simeq T_k - C$$

$$\Delta \phi \simeq \frac{2 \pi}{M}$$

$$\left(3\right)$$

where $\langle \cdot \rangle$ is the fractional part operator and $\sigma$ is the additive noise standard deviation which should be estimated prior to apply the SHT. Notice that (3) is the summation of two positive terms. The first one is due to lack of knowledge about the sinewave phase and is responsible for oscillations in the behavior of $\sigma_{\hat{T}_k}^2$ by varying the transition level $T_k$. Conversely, the second one takes into account the input noise contribution. Since the first term vanishes as $1/M^2$, while the second one tends to zero as $1/M$, for values of $M$ used in practice, the transition level estimator variance becomes

$$\sigma_{\hat{T}_k}^2 \simeq \frac{1}{M} \sqrt{A^2 - T_k^2} \leq \frac{1,78}{M} \frac{A \sigma}{M}$$

$$A \geq |T_k|,$$

$$k = 1, \ldots, 2^N - 1.$$  \hspace{1cm} (4)

In order to compare the variance of the SHT with the minimal achievable variance of any unbiased estimator of the ADC decision thresholds, the evaluation of the related CRLB when the ADC is stimulated by a noisy sinewave is of interest. In this regard, several simulations were carried out for different values of both the ADC transition levels and the AWGN standard deviation $\sigma$; only some meaningful results are reported in the following. Fig. 1(a) and (b) shows how the SHT variance $\sigma_{\hat{T}_k}^2$ and the CRLB depend on the transition level $T_k$ to be estimated when the ADC is fed by a sinewave with an amplitude that is 5% greater than the converter full scale, FS. In both figures, obtained for noise standard deviations $\sigma$ of 0.001 FS and 0.05 FS respectively, the case when $M = 9$ samples is considered. This small value of the number of samples has been chosen in order to analyze the effect of the sinewave phase.

Theoretical curves for the SHT estimator variance shown in Fig. 1(a) and (b) have been derived through (3), while simulation results for the CRLB have been obtained by estimating the partial derivatives of the Fisher information matrix using Monte Carlo methods based on $10^5$ data records [6]–[8]. Conversely, the theoretical plots for the CRLB have been obtained by using (B.13) reported in Appendix B, where (B.7) has been evaluated by means of numerical integration. It can be seen that simulation data show a very good agreement with theoretical results. It is worth noticing that noise smoothes the behavior of the CRLB curves. In fact, the CRLB oscillations are due to the lack of knowledge about the phase of the stimulus sinewave, whose effect tends to become negligible when the noise level increases [2].

In Fig. 1(b), the asymptotic expression (4) is also shown. Notice that the local minima of both the $\sigma_{\hat{T}_k}^2$ and the CRLB coincide with this expression. Thus, since for the large values of
Moreover, the code bin widths are estimated by subtracting consecutive transition voltages, that is

\[ \hat{W}_k = \hat{I}_{k+1} - \hat{I}_k, \quad k = 1, \ldots, 2^N - 2 \]

(6)

from which, the related DNL is

\[ \text{DNL}_k = \frac{\hat{W}_k}{Q} = \frac{\text{INL}_k - \text{INL}_{k+1}}{Q}, \quad k = 1, \ldots, 2^N - 2. \]

(7)

Notice that, since gain and offset errors in (5) can be estimated with high accuracy, an upper bound to the variances of the INLs can easily be derived by (4). Moreover, when the standard deviation \( \sigma \) of input noise is small compared to the ideal code bin width \( Q \), the ADC transition level estimators are almost uncorrelated and we have \( \sigma_{\hat{W}_k}^2 \cong \sigma_{\hat{I}_k}^2 + \sigma_{\hat{I}_{k+1}}^2 \) [4]. Thus, the variances of the DNL estimators are almost equal to twice the value given by (4) for the \( k \)-th threshold level. Conversely, when \( \sigma/Q \) is not small, both simulation and experimental results show that \( \sigma_{\hat{W}_k}^2 < 2\sigma_{\hat{I}_k}^2 \) due to the positive correlation between the estimators of adjacent decision thresholds.

D. SHT Experimental Validation

In order to validate (4), an experimental test was carried out on a 6023E 12-bit National Instruments’ data acquisition board (DAQ), operating in the \( \pm 10 \) V range at a maximum sampling rate of 200 ksample/s. Since the nominal INL and DNL of the 12-bit ADC are lower than 0.5 LSB, by discarding the two least significant bits, the board can be assumed to operate as an ideal 10-bit ADC. The DAQ stimulus was a 10-Hz sinewave with an amplitude \( A \) of 10.5 V, obtained from a Stanford Research DS360 generator, affected by an AWGN with standard deviation \( \sigma = 10 \) mV, obtained using an Agilent 33220A generator. Notice that, while the white noise generator has a nominal bandwidth \( B_1 \) of 10 MHz, the DAQ 3-dB input bandwidth \( B_2 \), measured according to [2], is about 609 kHz. Thus, considering \( B_1 \) and \( B_2 \) as sufficiently accurate estimates of the corresponding filter equivalent noise bandwidths, the desired value of \( \sigma \) at the input of the DAQ can be achieved by setting the output of the noise generator 33220A to a standard deviation \( \sigma \sqrt{B_1/B_2} \), that is about 40.5 mV. The SHT was repeated 40 000 times, using records of \( 2 \cdot 10^4 \) samples each. By fitting the experimental data to the theoretical curve (4), using a least-squares approach, \( \sigma \) and \( A \) have been estimated as 10.9 mV and 10.4 V respectively. Notice that the input noise variance was very close to the expected one, even though in the evaluation of \( \sigma \), both the noise generator and the DAQ frequency responses has been modeled as ideal low-pass filters. As shown by Fig. 2, which reports both the variance of the estimated transition voltages and (4) as a function of the code index, using the estimated values for \( \sigma \) and \( A \) leads to a very good agreement between theoretical and experimental data. In fact, the maximum relative error is less than 8%.

The experimental data were also used to evaluate the variance associated to the code bin widths estimator. Fig. 3(a) reports the measured code width variance \( \sigma_{\hat{W}_k}^2 \) as a function of the code index \( k \), together with the curve obtained by doubling the related transition voltage estimator variance as given by (4).
The ADC transition voltage estimators result corresponds to a 95% coverage level mV, in order to control the known mV, by a coverage factor and . It is determined by will be discussed. can easily be made negligible which between ad-

Since , the ADC transition voltage estimators result positively correlated. The correlation coefficient between adjacent threshold estimators, can be estimated using the relationship , by recalling that , by , and . The achieved results are reported in Fig. 3(b), which show that assumes small values. In fact, in the considered case it is and the transition level estimators are almost uncorrelated as expected [4].

III. SHT DESIGN

Designing the SHT requires the value selection of the input sinewave amplitude, offset, and frequency, and the choice of the number of collected samples , in order to control the known uncertainty sources and achieve the desired accuracy. In this section, test design criteria will be given, based on both the statistical properties of the SHT estimators and on the characteristics of the testing equipment.

As stated in Section II-B, the sinewave frequency should be chosen according to (2). However, the value of external or internal ADC sampling clock frequency is not perfectly known, also because of jitter phenomena. Moreover, further uncertainty sources are the limited amplitude accuracy and resolution of the waveform generator and the ADC gain and offset.

Usually, the major test uncertainties are due to the input AWGN. It introduces on the measurement results both a bias term, which can be reduced by overdriving the ADC, and zero mean random deviations, which can be made lower by increasing . It is worth of notice that the bias term cannot be accurately estimated and corrected. However, by properly overdriving the A/D converter the corresponding expanded uncertainties and can easily be made negligible with respect to the corresponding ones and due to random contributions (see Appendix A). According to [10], the expanded uncertainty represents the half length of the coverage interval in which a measurement result is assumed to fall with a specified coverage probability . It is determined by multiplying the estimated standard deviation of the measurement (standard uncertainty ) by a coverage factor () which depends both on the statistical distribution of the measurement results and on the coverage probability. For instance, for normal distributed data, which corresponds to a 95% coverage level [10]. In the Sections III-A and B, test design criteria useful for the selection of both the sinewave amplitude and the number of samples will be discussed.

A. Selection of the Sinewave Amplitude

The selected sinewave amplitude should ensure that all of the ADC codes are properly excited, and that the estimator bias due to the input AWGN is made negligible. The first issue can be addressed by keeping into account both the ADC characteristics and the waveform generator accuracy according to a worst case approach. In particular, the ADC under test is characterized by both gain and offset deviations, which affect the actual ADC input range, that is the difference between the last and first
transition voltages. During the test development phase, an upper bound for the uncertainty introduced by each of these factors has to be estimated a priori in order to ensure a proper ADC excitation. Then, after the test is carried out, it should be verified that the measured gain and offset deviations do not exceed the a priori assumed values. Should such a check fail, larger values of gain and offset deviations should be assumed, and the test should be repeated. Obviously, this procedure has to be carried out during the development phase of the test, that is, when choosing the proper test parameters. During production, the proper sinewave amplitude has already been chosen, and the test does not need to be repeated.

Notice that the ADC gain $G$ is the multiplicative factor relating the real transition voltages to the ideal ones [2]. Thus, if $U_{rG}$ represents the a priori relative expanded uncertainty for the ADC gain, the minimum input sinewave amplitude that ensures with high probability the excitation of all ADC output codes is $FS/G(1-U_{rG})$. Similarly, both the sinewave generator offset and the ADC offset may directly displace the transition voltages, and have to be taken into account. In particular, if $U_A$ and $U_C$ represent the related a priori expanded uncertainties respectively, the minimum sinewave amplitude $A_0$ which guarantees the excitation of all the ADC codes is given by

$$A_0 = \frac{FS}{(1-U_{rG})} + U_C + U_A$$

where the nominal gain $G$ has been assumed equal to one, and the uncertainties have been composed by considering the worst-case condition.

In order to guarantee with high confidence that the input AWGN does not introduce a significant bias in the estimated INLs and DNLs when the input voltage is near the first and last transition voltages, the following specifications apply for the sinewave amplitude (see Appendix A):

$$A \geq A_0 + V_{OD}$$

in which

$$V_{OD} = \max(V_{OD,INL}, V_{OD,DNL})$$

where $V_{OD,INL}$ is the target INL expanded uncertainty, expressed in LSBs, and $V_{OD,DNL}$ is the relative expanded uncertainty of the DNL estimator, that is, it is expressed as a fraction of the measured DNL value.

Notice that, if the noise standard deviation is lower than 1% of the sinewave amplitude, as often occurs in practice, for values of the maximum allowable INL uncertainty exceeding $\sigma/(5Q)$ and DNL uncertainty exceeding 65% there is no need to use overdrive (see Appendix A).

### B. Selection of the Number of Samples

Once the sinewave amplitude $A$ has been determined, the standard uncertainties of INL and DNL estimators due to the random contribution of the input AWGN, can be controlled by varying $M$. In fact, by properly rearranging (4), the minimum number of samples to acquire for achieving target values $u_{R,INL}$ and $u_{R,DNL}$, expressed in Least Significant Bit (LSB) units, for standard INL and DNL uncertainties is

$$M \geq \max \left( \frac{1}{U_{R,INL}} , \frac{2}{U_{R,DNL}} \right) \cdot 1.78A \sigma \left( \frac{K_\nu}{Q} \right)^2$$

$$U_{R,INL} = K_\nu u_{R,INL} , \quad U_{R,DNL} = K_\nu u_{R,DNL}$$

where $U_{R,INL}$ and $U_{R,DNL}$ are the corresponding expanded uncertainties and $K_\nu$ is the coverage factor corresponding to a coverage level $\nu$.

### IV. Example of SHT Implementation

To illustrate the use of the procedure described in Section II, a SHT was performed on a National Instruments 6023E 12-bit DAQ, operating in the $\pm10$ V input range at its maximum sampling rate of 200 ksample/s. The sinusoidal function generator requirements were an SFDR higher than 74 dB, because of the 12-bit ADC resolution, and an output range reasonably higher than the 10-V DAQ full scale, in order to be capable of properly overdriving the ADC under test. Thus, a Stanford DS360 Function Generator was used to generate a 200 Hz sinusoidal stimulus. Notice that, in order to characterize the dynamic behavior of the DAQ, the SHT should be repeated for different and higher sinewave frequencies. The function generator specifications do not provide information about the SFDR, but state the total harmonic distortion (THD) to be better than 105 dB [2]. Thus, by assuming that the intermodulation terms were negligible, the SFDR was assumed to be higher than the required 74 dB.

The function generator amplitude uncertainty was stated as being lower than 1% of the nominal amplitude, and the datasheets show that the effect of finite resolution was negligible in the considered case. As the sinusoid amplitude $A$ is about 10 V, this corresponds to an expanded uncertainty $U_A \cong 0.10$ V.

Furthermore, the uncertainties related to the ADC gain and offset were estimated from the DAQ specifications as $U_{rG} = 2.75\%$ and $U_C = 28$ mV when the allowed operating conditions are satisfied. Thus, by applying (8), we obtained $A_0 \cong 10.4$ V. To determine the required overdrive $V_{OD}$, the input AWGN standard deviation has been measured according to [2], estimating a value of 8 mV. Then, by assuming as target uncertainties due to the systematic effect of input noise $U_{B,INL} = 0.2$ LSB and $U_{B,DNL} = 10\%$, (10)-(12) lead to $V_{OD} \cong 20$ mV, which was negligible with respect to $A_0$. Thus, according to (10), the chosen sinusoid amplitude was $A \cong 10.4$ V.

As for our DAQ, the quantization step $Q$ is about 5 mV, by assuming as target expanded uncertainties due to random effect of the input noise $U_{R,INL} = 0.2$ LSB and $U_{R,DNL} = 0.3$ LSB, with a coverage factor $K_\nu = 3$, the minimum number of samples provided by (13) is $M \geq 1.5 \cdot 10^9$. In order to satisfy (2)
and using the procedure suggested in [2], for an input frequency of 200 Hz \( (J = 1685) \), \( M \) was selected to be 1 684 999.

Once the test parameters are determined, the SHT was carried out and the ADC gain and offset deviations were computed according to [2], leading to values of \( U_{R_{G}} = 0.14\% \) and \( U_{C} = -2.8 \) mV respectively. These uncertainties are lower than the \( a \) priori values (2.75% and 28 mV) thus validating the consistency of the test.

Finally, INL and DNL, were computed from (5) and (7), leading to the results of Fig. 4. The standard uncertainties \( u_{R_{INL}} \) and \( u_{R_{DNL}} \) have been evaluated from the experimental data according to [10], obtaining \( u_{R_{INL}} = 0.06 \) LSB and \( u_{R_{DNL}} = 0.016 \) LSB. By using \( K_{\sigma} = 3 \), the corresponding expanded uncertainties are \( U_{R_{INL}} = 0.2 \) LSB and \( U_{R_{DNL}} = 0.05 \) LSB, which are in good agreement with the target uncertainties \( U_{B_{R_{INL}}} \) and \( U_{B_{R_{DNL}}} \). Moreover, by substituting \( U_{R_{G}} \) and \( U_{C} \) in (8), it results an a-posteriori sinewave amplitude \( A_{0} \approx 10.1 \) V. Thus, according to (9), the actual test overdrive \( V_{OD} \) was \( A - A_{0} = 10.4 - 10.1 = 0.3 \) V. By replacing such a value in (11) and (12), we confirm that the effect of input AWGN on estimator bias was negligible. Thus, the proposed testing procedure effectively satisfied the target uncertainties requirements.

V. CONCLUSION

The performances and the efficiency of the SHT, have been discussed and analyzed in this paper. The characterization of the test performance has been improved by removing some commonly adopted approximations and because controlled experiments have been done to validate models and published expressions. The statistical efficiency of the SHT has been evaluated by comparing the estimator variance to the corresponding CRLB, computed both theoretically and by means of simulations. Furthermore, precise directions have been given on how to choose the proper amount of ADC overdrive. Finally, the results suggest that the accuracy of the SHT is asymptotically optimal also when Gaussian white noise is superimposed to the sinusoidal stimulus.

APPENDIX A
DERIVATION OF PROPOSED OVERDRIVE

In the literature the sampled voltage amplitude distribution is computed by convolving the amplitude distribution of a perfect sinusoidal signal with the probability density function (pdf) of Gaussian noise (see [4, eq. (31)]). The probability distribution of the estimated transition voltages is then approximated using a Taylor expansion leading, after some simplification, to [4, eq. (33)]. The overdrive is then determined so that the maximum deviation for the INL is less than \( B/4 \) LSBs where \( B \) is specified by the user

\[
V_{OD_{INL}} = \frac{\sigma^2}{QB}.
\]

Because of the approximations used, the actual error is 28% greater than \( B/4 \) when using \( V_{OD_{INL}} \) given by (A.1) for values of \( V_{OD_{INL}} \) lower than \( 2\sigma [4] \), leading to expression (9) proposed in [4] and also used in [2, eq. (10)]. Here, we suggest three changes to it, leading to (11).

- Instead of defining the target expanded uncertainty for the INL as a fraction of 1/4 LSB (B) we define it as a fraction of 1 LSB. So \( U_{B_{INL}} = B/4 \).
- We calculated numerically the actual INL deviation for a wider range of values of \( \sigma \) than the one considered in [4] and found that under such condition it is in the worst case 28% higher than the value given by the approximation leading to (A.1) for any value of \( V_{OD_{INL}} \), not just for values higher than \( 2\sigma \) as reported in [4]. So we propose to replace the factor \( 2\sigma \) with a multiplying factor 1.3, as shown in (11).
- In Fig. 5 we represent the results of the numerical calculation of the transition voltage estimation error. It can be observed that the error is always lower than \( \sigma/5 \) for \( \sigma \) lower than 1% of the sinusoidal amplitude (A). Consequently, overdrive should be used only when \( U_{B_{INL}} \) is lower than \( \sigma/5 \). For higher values of target uncertainty, there is no need to use overdrive because the error will never exceed \( U_{B_{INL}} \), as shown in Fig. 5 for \( A = 1(A = V_{R}/2) \).
### APPENDIX B

**DERIVATION OF CRLB FOR THE ADC DECISION THRESHOLDS WHEN THE INPUT SIGNAL IS A SINEWAVE AFFECTED BY ADDITIVE GAUSSIAN NOISE**

Let us indicate with $T$ the vector of the $2^N - 1$ ADC transition levels to be estimated, and let us define $Y$ as a vector random variable (rv) whose possible realizations belong to the space of the experimental outcomes. Furthermore, let us assume that $t_T$ is a scalar statistic of the sample space associated to the ADC output, expressed as a function of the unknown ADC transition levels. The CRLB associated to the variance of an estimator $t'$ of $t_T$ with a bias $b_{t_T} = E[t'] - t_T$ is given by

$$CRLB = [\nabla t_T + \nabla b_{t_T}]^T F^{-1} [\nabla t_T + \nabla b_{t_T}]$$  \hspace{1cm} (B.1)

where $\nabla$ is the gradient operator, that is

$$\nabla t_T = [\partial t_T / \partial T_0, \partial t_T / \partial T_1, \ldots, \partial t_T / \partial T_{N-1}]$$

$$\nabla b_T = [\partial b_T / \partial T_0, \partial b_T / \partial T_1, \ldots, \partial b_T / \partial T_{N-1}]$$

and $F$ is the Fisher information matrix [5]. As this analysis is focused on the CRLB associated to the estimation of the $i$th transition level $T_i$, we have $t_T = T_i$ and

$$\frac{\partial t_T}{\partial T_j} = \begin{cases} 1, & i = j, \\ 0, & i \neq j. \end{cases}$$  \hspace{1cm} (B.2)

Let us define $P\{Y = Y; T\}$ as the probability of occurrence of a given realization $Y$ in the sample space of the ADC output. The Fisher information matrix can be expressed as [7]

$$F = E[\nabla \log P\{Y = Y; T\}] [\nabla \log P\{Y = Y; T\}]^T$$  \hspace{1cm} (B.3)

where $E[\cdot]$ is the expected value operator. By expanding (B.3), the elements of the Fisher information matrix may be obtained as

$$F_{n,m} = \sum_{Y} \frac{1}{P\{Y = Y; T\}} \frac{\partial P\{Y = Y; T\}}{\partial T_n} \frac{\partial P\{Y = Y; T\}}{\partial T_m},$$

where

$$n = 1, \ldots, 2^N - 1; \quad m = 1, \ldots, 2^N - 1$$  \hspace{1cm} (B.4)

where the summation is performed over all possible realizations $Y$ for which $P\{Y = Y; T\} \neq 0$.

In the following subsections, the CRLB on the estimation of the ADC transition level will be discussed for an ADC stimulus consisting of a sinewave corrupted by additive white Gaussian noise. Both unbiased and biased estimators will be considered.

**A. Unbiased Estimators**

Let us assume that the ADC stimulus is expressed by the following:

$$x_n = A \sin \left( 2\pi \frac{L}{M} n + \phi \right) + \eta_n, \quad n = 0, \ldots, M - 1$$  \hspace{1cm} (B.5)

where $M$ is the record length, $L$ and $M$ are mutually prime numbers, $\phi$ is the initial record phase, which is uniformly distributed in $(0, 2\pi)$ when different records are considered, and $\eta_n$ is a zero mean Gaussian white noise with standard deviation $\sigma$. Let us define $X = [x_0 \cdots x_{M-1}]$ as the vector of random variables that models the samples at the ADC input. Similarly,
let us define $Y = [y_0 \ldots y_{2^N-1}]$ as the vector of the corresponding ADC output codes. If $2^N - 1$ is the number of ADC thresholds and $2^N$ the number of output codes, $Y$ may assume $(2^N)^M$ different values. In a single record of $M$ samples, $\phi$ is constant, so the ADC input (B.5) is a sequence of normally distributed and mutually independent random variables, with mean values $E[x_n] = A \sin(2\pi L/n + \phi)$. Hence, by noticing that the considered ADC transfer function is memoryless, the probability of occurrence of a given ADC output sequence is given by the product of the marginal probabilities of occurrence of the individual samples, that is

$$P\{Y = Y; T\} = E[P\{Y = Y \mid \phi; T\}]$$

$$= \frac{1}{2\pi} \int_0^{2\pi} \prod_{j=0}^{M-1} P_j(y_k \mid \phi; T) d\phi \quad (B.6)$$

where $P_j(y_k \mid \phi; T)$ is the probability that the $j$th sample of the ADC output equals a given output code $y_k$, e.g., $P_j(y_k \mid \phi; T) = P\{T_i < y_j \leq T_{i+1}\}$ when $0 < i < 2^N - 1$, conditioned to the values assumed by the initial phase $\phi$, and where the outermost expectation is carried out over $\phi$. This operation, performed by integrating on the real axis the product of the marginal probabilities $P_j(y_k \mid \phi; T)$ and the phase probability density function, removes the dependence on $\phi$. Thus, the marginal probabilities can be easily defined in terms of the noise probability distribution function, giving (B.7) at the bottom of the page where $\Phi(\cdot)$ is the probability distribution function of a zero mean unity-variance Gaussian rv.

In order to derive the Fisher matrix, the partial derivatives of $P(Y = Y; T)$ with respect to the transitions levels $T_i$ are needed. To this extent, it can be observed that $P_j(y_k \mid \phi; T)$ is differentiable with respect to any $T_i$. Hence, according to (11, Th. 4.1) it is possible to differentiate under the integral sign, as expressed by the following:

$$\frac{\partial}{\partial T_i} P\{Y = Y; T\} = \frac{1}{2\pi} \int_0^{2\pi} \prod_{j=0}^{M-1} P_j(y_k \mid \phi; T) d\phi \cdot \frac{\partial}{\partial T_i}$$

$$\int_0^{2\pi} \prod_{j=0}^{M-1} P_j(y_k \mid \phi; T) d\phi = \frac{1}{2\pi} \int_0^{2\pi} \prod_{j=0}^{M-1} P_j(y_k \mid \phi; T) d\phi \times$$

$$h = 1, \ldots, 2^N - 1; \quad i = 0, \ldots, 2^N - 1 \quad (B.8)$$

where the derivative of the product of the marginal probabilities can be rewritten as

$$\frac{\partial}{\partial T_i} \prod_{j=0}^{M-1} P_j(y_k \mid \phi; T) = \sum_{j=0}^{M-1} \left[ \left( \frac{\partial}{\partial T_i} P_j(y_k \mid \phi; T) \right) \prod_{k \neq j} P_k(y_k \mid \phi; T) \right]$$

$$i = 0, \ldots, 2^N - 1; \quad h = 1, \ldots, 2^N - 1; \quad k = 0, \ldots, M - 1. \quad (B.9)$$

By applying (B.8), the derivatives of the marginal probabilities are expressed by (B.10) at the bottom of the page where $\varphi(\cdot)$ is the pdf of a Gaussian zero mean unity-variance rv. Finally, (B.6) is evaluated using (B.8), and, by substituting equations (B.10) in (B.9), the Fisher information matrix (B.4) can be derived using (B.6) and (B.8).

B. Biased Estimators

When biased estimators are considered, the CRLB depends on how the bias varies with the ADC transition levels. In particular, for SHT estimators we have [3]

$$b_{TK} = E[x_k] - t_{TK} = \frac{\sigma^2}{2} \frac{T_k}{A^2 - T_k}, \quad T_k = T_k - C, \quad k = 1, \ldots, 2^N - 1 \quad (B.11)$$

where $b_{TK}$ is the $k$th element of $b_T$, that is the bias of the $k$th ADC transition level estimator. It follows that the partial derivatives of $b_T$ are given by

$$\frac{\partial b_{T_i}}{\partial T_j} = \begin{cases} \frac{\sigma^2}{2} \frac{A^2 + T_i^2}{(A^2 - T_j)^2}, & i = j, \quad i, j = 1, \ldots, 2^N - 1, \\ 0, & i \neq j \end{cases} \quad (B.12)$$

In particular, when a single-bit ADC is considered, from (B.4) we have

$$\text{CRLB} = \left( 1 + \frac{\sigma^2}{2} \frac{A^2 + \text{var}^2}{(A^2 - T_0^2)^2} \right)^{-1}. \quad (B.13)$$

Notice that, when a small overdrive is used and the Gaussian noise power is not negligible with respect to the sinewave one,
the transition voltage estimator variance $\sigma_{2V}^2$ provided by (4) may be lower than (B.13) when $T_k$ approaches $A$. This behavior can be explained by observing that under such conditions the accuracy of (3) is greatly reduced [5], [12].

ACKNOWLEDGMENT

The authors are grateful to an unknown reviewer whose comments helped to rewrite the paper in its present form.

REFERENCES


Francisco André Corrêa Alegria was born in Lisbon, Portugal, on July 2, 1972. He received the Diploma, Master’s degree, and Ph.D. degree in electrical engineering and computers from Instituto Superior Técnico (IST), Technical University of Lisbon, Portugal, in 1995, 1997, and 2002, respectively.

He has been a Member of the teaching and research staff of IST, Technical University of Lisbon, since 1997. He is a member of the Instrumentation and Measurement research line at the Instituto de Telecomunicações, Technical University of Lisbon, where he has been since 1994. His current research interests include analog-to-digital characterization techniques, automatic measurement systems, and computer vision.

António Manuel da Cruz Serra was born in Coimbra, Portugal, on December 17, 1956. He received the Diploma in electrotechnical engineering from the University of Oporto, Oporto, Portugal, in 1978 and the Master’s and Ph.D. degrees in electrical engineering and computers from Instituto Superior Técnico (IST), Technical University of Lisbon, Lisbon, Portugal, in 1985 and 1992, respectively.

He is Full Professor of Instrumentation and Measurement at IST, Technical University of Lisbon, where he has been a member of the teaching and research staff since 1978. He is a Member of the Instrumentation and Measurement research line at the Instituto de Telecomunicações, Technical University of Lisbon, where he has been since 1994. His current research interests include electrical measurements, analog–digital conversion characterization techniques, and automatic measurement systems.

Dario Petri received the Laurea and the Ph.D. degrees in electronic engineering from the University of Padova, Padova, Italy, in 1986 and 1990, respectively.

From 1990 to 1992, he was at the Dipartimento di Ingegneria Elettronica e Informatica of the same university as a Research Fellow. Then, he joined the Dipartimento di Ingegneria Elettronica e dell’Informazione, the University of Perugia, Perugia, Italy, in 1992 as an Associate Professor. From 1999 to 2002, he was a Full Professor of Electronic Instrumentation in the same Department and the Chairperson of undergraduate and graduate degree study programs in Information Engineering. In 2002, he joined the Dipartimento di Informatica e Telecomunicazioni of the University of Trento, Trento, Italy, where he is currently the Dean of the International Ph.D. School in Information and Communication Technologies. His research activities are in the general areas of measurement science and technology, with particular interest to: data acquisition system design and testing, digital electronic system design, system characterization and performances, application of digital signal processing and statistical parameter estimation methods to measurement problems. He is author and coauthor of more than 100 papers published in international journals or in proceedings of international congresses.