A Total Variation based Denoising Algorithm for 3D Ultrasound

João M. R. Sanches
Instituto de Sistemas e Robótica
Instituto Superior Técnico
Lisbon, Portugal

Abstract
This paper presents an algorithm for denoising a three dimensional image, under the Rayleigh distributed multiplicative noise model, which is the observational model for Ultrasound imaging. The proposed method performs a variable splitting to introduce an auxiliary variable to serve as the argument of the 3D total variation term. This leads to two problems involving the data fidelity term and the regularizer, which are solved alternatively leading to an instance of the Alternating Direction Method of Multipliers (ADMM) method, for which convergence is guaranteed. This framework offers a simple way to apply the TV regularizer as at each iteration, one needs to minimize the sum of a TV term and a least squares term, as against the Rayleigh penalty function. This framework can further be extended to decompose the image and speckle.

1 Introduction
Ultrasound (US) has emerged as a popular medical imaging modality in a number of medical imaging applications because of its lower cost, wide reach, flexibility, lack of radiation, and intra-operability. In Brightness mode (B-mode) 2D US, a two-dimensional image is acquired by a linear array of transducers simultaneously scanning a plane through the body. However, 2D imaging only allows views in the same plane as the US acquisition is performed. Another limitation is that the 2D US image represents a thin plane at some arbitrary angle in the body and it may be difficult to localize the image plane and reproduce it later for follow-up [6].

Because of these limitations of 2D US, three dimensional (3D) US imaging is being increasingly used for characterizing certain diseases such as carotid atherosclerosis, which is a leading cause of death or disability. This often requires the 3D volume of the carotid artery to be reconstructed from a series of 2D slices. The slices can be acquired mechanically wherein the probe position is controlled by a motor which sweeps it over the volume in a predetermined manner, or freehand wherein the user can manually position and orient the probe. Due to the impracticability of slicing over the entire volume, voxel values of the 3D object are available for only a small set of the 3D positions.

1.1 Problem Formulation
We assume that the dimensions of the volume, that is, the number of voxels is \( L \times M \times N \). We will represent the volume as a vector in lexicographic ordering, \( \mathbf{x} \in \mathbb{R}^n \), where \( n = L \times M \times N \) is the number of voxels. Assuming linear mechanical scanning along the height, we denote the set of observed 2-D slices as \( \{S_i \in \mathbb{R}^{M \times N}, i = 1, \ldots, n_i \} \). The acquired 2-D slices are of size \( M \times N \), and the number of slices is \( n_i < L \). Hence the number of observed voxels is \( m = n_i MN \). Knowing the scanning pattern over the volume, we obtain the position coordinates of each pixel in the observed slices. We can therefore define an observed vector \( \mathbf{y} \in \mathbb{R}^m \), which contains the voxel values from the observed slices after mapping to the volume and ordering in lexicographic order. In the case of scanning along a linear line, the slices can be simply stacked. In the more general case where the acquisition is random and freehand, this sampling pattern can be applied, knowing the positions and orientations of the acquisition patterns.

Knowing the coordinates of the pixels of the 2-D slices, we can define a 3-D binary array which is 1 at the voxels corresponding to pixels in observed slices and 0 otherwise. This leads to a linear observation model for modeling the acquisition of slices from the volume

\[
\mathbf{y} = \mathbf{A} \mathbf{x},
\]  

where the linear operator \( \mathbf{A} \in \{0,1\}^{m \times n} \) multiplies the vector representing the 3-D volume with the binary mask, and discards the voxels that are not sampled. It is essentially the \( n \times n \) identity matrix with \( n - m \) rows removed. For a denoising problem, it is equal to the identity matrix \( \mathbf{A} = \mathbf{I} \), which is the case when all voxel values are observed.

In US images, the noise is multiplicative and is called speckle,

\[
\mathbf{y} = (\mathbf{A} \mathbf{x}) \eta,
\]  

where \( \eta \) is the speckle field that is multiplied element-wise with the observed volume. For the Radio Frequency envelope images, it is assumed that the observed image follows Rayleigh statistics, with the likelihood

\[
p(\mathbf{y}|\mathbf{x}) = \prod_{i=1}^{m} \frac{y_i}{\|\mathbf{A} \mathbf{x}\|_1} \exp \left( -\frac{y_i^2}{2\|\mathbf{A} \mathbf{x}\|_1} \right),
\]  

where \( \|\mathbf{A} \mathbf{x}\|_1 \) is the \( p^{th} \) element of the sampled vector \( \mathbf{A} \mathbf{x} \), and \( y_i \) is the \( p^{th} \) element of the noisy observed vector \( \mathbf{y} \).

To formulate the optimization problem to estimate the volume \( \mathbf{x} \), given \( \mathbf{y} \) with the Rayleigh multiplicative model (3), we apply a logarithmic transformation,

\[
\mathbf{f} = \log(\mathbf{x}),
\]  

and then use the convex data fidelity term from [8], leading to the total variation (TV) regularized convex problem

\[
\min_{\mathbf{f}} \sum_i \left( \frac{\mathbf{y}_i^2}{2} e^{-f_i} + f_i \right) + \alpha TV(\mathbf{f}),
\]  

where as before, \( f_i \) represents a voxel from the vector representation of the volume \( \mathbf{f} \), indexed by the coordinates \( (i_x, i_y, i_z) \), \( \alpha > 0 \) is the regularization parameter, and the regularizer function is the 3D TV function which is given by

\[
TV(\mathbf{f}) = \sum_i \left( (f_{i_x, i_y, i_z} - f_{i_x, i_y, i_z-1})^2 + (f_{i_x, i_y, i_z} - f_{i_x, i_y, i_z-1})^2 + \cdots + (f_{i_x, i_y, i_z} - f_{i_x, i_y, i_z-1})^2 \right)^{\frac{1}{2}}.
\]  

Although there exist fast solvers for the sum of a quadratic data fidelity term and a TV term [3, 7], we cannot apply them to the multiplicative and Rayleigh noise model.

1.2 Contributions
This work proposes an algorithm to solve the denoising problem (5), based on the Augmented Lagrangian (AL)/Alternating Direction Method of Multipliers (ADMM) framework [5] which decompose the problem into a series of simpler problems, are fast, and have guaranteed convergence conditions.

2 Proposed Method
We find the solution of problem (5) by using an approach based on the AL/ADMM framework. In [2], an ADMM based method was presented for solving a problem similar to (5) but without logarithmic compression, for the case of Synthetic Aperture Radar (SAR) imaging. We perform a variable splitting [4] and introduce an auxiliary variable \( \mathbf{u} \) to serve as the argument of the TV term, with the constraint \( \mathbf{f} = \mathbf{u} \). This leads to the constrained optimization problem,

\[
\min_{\mathbf{f}, \mathbf{u}} \sum_i \left( \frac{\mathbf{y}_i^2}{2} e^{-f_i} + f_i \right) + \alpha TV(\mathbf{u}), \quad \text{subject to} \quad \mathbf{f} = \mathbf{u}.
\]
Using the augmented Lagrangian, this problem can be shown to be equivalent to the minimization problem,
\[
\min_{f,u} \sum_i \left( \frac{\sigma_i^2}{2} e^{-f_i} + f_i \right) + \alpha TV(u) + \frac{\mu}{2} ||f - u - d||^2_2, \tag{8}
\]
where \(\mu \geq 0\) is called the AL penalty parameter, and \(d\) is the so-called Bregman update vector [7]. The AL algorithm iterates between minimizing the objective function in (8) with respect to \(f\) and \(u\), leading to a Gauss-Seidel process (for more details, see [1]) which at iteration \(k\) is summarized as,
\[
\begin{align*}
    u^{k+1} &= \arg\min_{u} \alpha TV(u) + \frac{\mu}{2} ||f - u - d||^2_2, \tag{9}
    \\
    f^{k+1} &= \arg\min_{f} \sum_i \left( \frac{\sigma_i^2}{2} e^{-f_i} + f_i \right) + \frac{\mu}{2} ||f - u^{k+1} - d||^2_2, \tag{10}
    \\
    d^{k+1} &= d^k + u^{k+1} - f^{k+1}.
\end{align*}
\]

Problem (9) is a quadratic denoising problem and is solved using a 3D implementation of Chambolle’s algorithm [3]. The objective function in (10) is separable for each voxel \(f_i\), and can be decomposed into \(n\) problems,
\[
f_i^{k+1} = \arg\min_{f_i} \frac{\sigma_i^2}{2} e^{-f_i} + f_i + \frac{\mu}{2} (f_i - u_i^{k+1} - d_i^k)^2, \tag{11}
\]
which can be solved efficiently using a few iterations of Newton’s method [2]. Even though (9) and (10) are not solved exactly, the convergence conditions for ADMM [5] only require that their error sequences decrease monotonically. The final estimate of the volume is \(\hat{X} = \exp(f)\).

### 3 Experimental Results

We first demonstrate the proposed method on a synthetic example, in which a synthetically generated cylinder is corrupted with speckle, and then for denoising real B-mode US images of a carotid artery, stacked to form a volume. In the synthetic example, the generated cylinder is the ground truth and the mean square error can be computed relative to it. All experiments were performed on MATLAB on an Ubuntu Linux based laptop, with the Intel i5 processor and 8 GB of RAM.

The synthetic cylinder is a volume of size 128 \(\times\) 128 \(\times\) 128 and consists of ones in the volume of the cylinder, and zeros elsewhere. A cross-section in the xy-plane is shown in Fig 1(a). After speckling, the corresponding slice from the noisy volume is shown in Fig 1(b). Fig 1(c) shows the corresponding slice after denoising using the proposed method, and Fig 1(d) shows the denoised cylinder in a 3D view. The mean square error (MSE) between the original synthetic volume \(x\) and the denoised volume \(\hat{x}\), was 0.003, and the CPU time taken was 55.64 seconds.

The US images of the carotid artery were acquired transversally over a region of length 8 cm. There were 60 slices of size 255 \(\times\) 256, each roughly corresponding to an area of 3.9 \(\times\) 4 sq.cm. Fig 2(a) shows a B-mode image (which has speckle), and Fig 2(b) shows the corresponding slice from the denoised volume, shown in Fig 2(c). The CPU time was 162.59 seconds.

### 4 Conclusions

We have proposed an ADMM based algorithm for denoising images with multiplicative speckle noise. Preliminary results show that the proposed method is accurate and computationally efficient. Extension for reconstruction with a sub-sampling of the slices, speckle estimation, and comparison with other solvers will be addressed in a future paper.

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### References


