ARMA estimation – a comparison study for fMRI

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Abstract

The paper investigates the effectiveness of four projection methods of finite response function (FIR) to infinite response function (IIR) filters. This is done in the scope of a linear physiological model developed for functional Magnetic Resonance Imaging (fMRI). Of the four methods the Steiglitz-Mcbride algorithm provided the best fit. A new method is also proposed where a regularization term is used to keep the poles of the IIR filter inside the unit circle in order to make it stable. Preliminary results are promising.

1. Introduction

The applications of Infinite Response Function (IIR) models, also known as autoregressive-moving average (ARMA) models have found great attention in a wide range of control and signal processing areas such as time series analysis, signal modeling, spectral estimation, system identification, etc. Autoregressive Moving average (ARMA) is a parametric based method of signal representation. A variety of methods for estimating the parameters of an ARMA model have been developed in the realm of statistical and engineering literature [1–10] suitable for problems in which the signal can be modeled by explicit known source functions with a few adjustable parameters.

Without loss of generality, the transfer function of an ARMA filter is given by

$$H(z) = \frac{B(z)}{A(z)} = \frac{\sum_{j=0}^{q} b_j z^{-j}}{\sum_{k=0}^{p} a_k z^{-k}}$$

(1)

with $a_0=1$. The expression defined in (1) is also called the transfer function of a causal stationary ARMA model of order $(p,q)$, and $\{a_k ; 1 \leq k \leq p\}$ and $\{b_l ; 1 \leq l \leq q\}$ are the respective AR and MA parameters of the ARMA model. In the case of real data, corrupted with noise or not perfectly following an ARMA model the following difference equation holds

$$y(n) = -\sum_{k=1}^{p} a_k y(n-k) + \sum_{l=0}^{q} b_l x(n-l) + \eta(n)$$

(2)

where $\eta(n) \sim N(0,\sigma^2)$ is a residue, $x(n)$ is the input sequence and $y(n)$ the output sequence. The estimation of $a = [a_1, a_2, ... a_p]$ and $b = [b_1, b_2, ... b_q]$, defining the ARMA model that best describes the output sequence $y = [y(1) \ y(2) \ ... \ y(N)]$ given the input sequence $x = [x(1) \ x(2) \ ... \ x(N)]$, is not an easy task where the stability of the model is a central issue. Non linear optimization techniques are required.

Different methods are described in the literature where optimal and sub-optimal solutions are provided. Optimal solutions, according to a given criterion, e.g. minimum square error (MSE) or maximum likelihood (ML), lead to highly demanding algorithms from a computational point of view and numerically unstable or slowly convergent. These approaches usually lead to sets of non linear equations.

Sub-optimal approaches, on the contrary, are very popular because they are usually formulated as linear optimization problems that are more stable from a numerical point of view with solutions close to the optimal one. In one of the most used approach the AR coefficients are first estimated and the MA coefficients are estimated in a second step from a signal previously filtered.

Functional Magnetic Resonance Imaging (fMRI) is a new and powerful method that is mainly used in the detection of the cerebral structures and regions involved in a particular task.

It is sensitized to the paramagnetic properties of deoxyhaemoglobin which concentration locally fluctuates in strong correlation with the physiological events of brain activity.

In previous works, we have developed an fMRI data analysis algorithm [11] supported on an ARMA model of the hemodynamic response function (HRF) [12], which considers physiological variables such as vascular response, and oxygen metabolic rate change to stimulus and accounts for neural demand and systemic feedback control of vascular response.

Here we aim at finding, among Prony [8], Shanks’s [9], Steiglitz-Mcbride [10] and a new one proposed by the authors, the one that present the best results in the parameter estimation of our ARMA model using...
2. Experimental Results

The sub-optimal algorithms selected for comparison, all have enjoyed considerable success and acceptance in several areas: Shanks’s methods [9], Prony’s method [8] and Steiglitz-McBride method [10]. Also, an in-development algorithm, called Stable ARMA Estimation with Regularization in the Poles Positions (SAERPP), is used in the comparison. In this method the AR coefficients are estimated by forcing the poles of the filter to be located inside the unit circle in order to make it stable. This algorithm is under development and more sophisticated priors are being designed.

The methods were compared using synthetic data obtained from real Hemodynamic Response Function fMRI observations. The physiologically based hemodynamic (PBH) model [15] used in the study is an ARMA(3,2) model of 3 poles and 2 zeros from which the observed signal corrupted with zero mean Additive White Gaussian Noise (AWGN) of 10dB, 5dB, 1dB and 0dB SNR is generated and used by the four algorithms to estimate the underlying model.

Table 1 – Comparative table results with the MSE and variance percentual values for the ARMA(3,2) estimation for each algorithm.

<table>
<thead>
<tr>
<th>SNR</th>
<th>Prony</th>
<th>Shanks</th>
<th>Steiglitz-McBride</th>
<th>SAERPP</th>
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</thead>
<tbody>
<tr>
<td></td>
<td>$\bar{\xi}$</td>
<td>$\sigma$</td>
<td>$\bar{\xi}$</td>
<td>$\sigma$</td>
</tr>
<tr>
<td>15 dB</td>
<td>3.30</td>
<td>0.32</td>
<td>2.41</td>
<td>0.42</td>
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<tr>
<td>10 dB</td>
<td>4.91</td>
<td>0.48</td>
<td>3.12</td>
<td>0.31</td>
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<tr>
<td>5 dB</td>
<td>5.49</td>
<td>0.55</td>
<td>4.70</td>
<td>0.48</td>
</tr>
<tr>
<td>1 dB</td>
<td>6.29</td>
<td>0.61</td>
<td>6.00</td>
<td>0.57</td>
</tr>
<tr>
<td>0 dB</td>
<td>6.79</td>
<td>0.67</td>
<td>5.48</td>
<td>0.54</td>
</tr>
<tr>
<td>-2 dB</td>
<td>6.57</td>
<td>0.65</td>
<td>6.41</td>
<td>0.63</td>
</tr>
<tr>
<td>T $10^{-3}$ s</td>
<td>0.2376</td>
<td>0.1961</td>
<td>0.7617</td>
<td>0.0364</td>
</tr>
<tr>
<td>T (%)</td>
<td>31.2%</td>
<td>25.7%</td>
<td>100%</td>
<td>44.1%</td>
</tr>
</tbody>
</table>

The performance of each method is assessed with Monte Carlo tests where the Euclidean norm of the error signal, $e_k = \| \tilde{y}_k(n) - y(n) \|$ is computed. $\tilde{y}_k(n)$ is the signal generated by the estimated model and $y(n)$ is the noiseless output of the true model. Also, processing time was measured and is presented in normalized form. Table 1 displays the Monte Carlo results, mean and variance, for different amounts of noise and for each tested method.

3. Conclusions

This paper presents a comparison between for methods used to estimate an ARMA model from a finite length response signal. Although this document currently presents a limited comparison, the pursuit of this work will develop into a more broad comparison study incorporating more estimation methods, extensive Monte-Carlo tests and several PBH ARMA models to be estimated.

With the present results Steiglitz-Mcbride provided the overall better fit, but at the expense of significant computing time. The other algorithms provided rather similar results with a slight advantage for Shanks’s method. Although SAERPP method proved the worst results, it is still in development and it is the only method restricting instability of the estimated ARMA filter. This is not shown in the example presented but is of crucial relevance in the aimed fMRI application.

10. References


Figure 1 – Example of the selected PBH ARMA(3,2) model estimation results for 10dB of SNR. ARMA impulse response $y(n)$ in black, $y(n)$ with AWGN in blue and $\hat{y}_k(n)$ estimation in red. Pole-Zero Maps in the bottom and Impulse Time Response at the top.