Output synchronization of heterogeneous LTI plants with event-triggered communication

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Abstract— **This paper proposes a control architecture to achieve output synchronization of a group of heterogeneous LTI plants. To each plant, we associate a local controller that comprises a dynamic output feedback controller and a reference generator. The local dynamic output feedback controller is designed such that the output of the plant tracks the output of the reference generator. The reference generator includes an internal state that should be synchronized across all reference generators. The decision to broadcast this synchronization state to its neighbors is done by employing an event-triggered communication protocol. It is shown that, with the proposed control architecture, the solutions of the closed-loop system are globally bounded and that there exist positive lower bounds for the inter-event intervals generated the by event-triggered mechanism associated with the broadcast of synchronization states. A self-triggered implementation of the proposed event-triggered communication protocol is also presented.**

I. Introduction

A recurrent task in problems regarding multi-vehicle systems is the need to synchronize the vehicles in some sense. Problems of this kind for linear time-invariant (LTI) plants include consensus (see, e.g., [1]–[3]) and synchronization (see, e.g., [4], [5]), to name a few. In this paper, we address the output synchronization problem for a group of heterogeneous LTI plants where the goal is to derive decentralized control laws capable of guarantee that the output of each plant converges to a certain reference signal of a particular class. Even if each plant tracks its own reference signal perfectly, due to different initial conditions, the plants are not guaranteed to converge to the same reference signal. Hence, communication among controllers of different plants is required in order to correct this misalignment.

Due to the digital nature of the control devices and of the communication network, an additional constraint on the controller design is that local feedback and communication can only occur at discrete time instants. The standard approach is to assume *a priori* that broadcasts are made periodically. However, in recent years, a different strategy as been gaining traction due to a flurry of theoretical developments. Known as event-triggered control, in this new approach control action or broadcast of data only occur when deemed necessary by some triggering condition, often dependent on the state of each plant. In the single plant case, an event-triggered controller operates as follows. An event detector is responsible for testing if a triggering condition (basically, a function of the plant's state) is true or false. If true, then the control input is updated. For more details regarding event-triggered control, the interested reader is referred to [6]–[10] for stabilization of a single plant and to [11], [12] for the multiple plant case. Note that in event-triggered control, the state or output of the plant must be constantly monitored which, depending on the application, may be infeasible. To avoid the need of constant monitoring, self-triggered control strategies are proposed in [13]–[17] where, instead of continuously testing a triggering condition, an event scheduler is responsible for computing when the next sampling event should occur. When there are several plants that need to communicate with each other, the event-triggered strategy is even more relevant since the communication medium is often shared by all plants, meaning that if each plant tries to access the communication network too often, soon successful communication will be impossible. Hence, by resorting to event-triggered control techniques, a communication protocol that avoids redundant broadcasts of information is derived.

In this paper, we address the problem of event-triggered output synchronization. To the best of our knowledge, this problem has not been addressed yet. The control architecture proposed is inspired by the work reported in [5] for output synchronization of heterogeneous LTI systems and in [18] for event-triggered consensus (the consensus problem is a particular type of synchronization problem where the reference signal is constant). Other examples of the application of event-triggered techniques to the problem of consensus that may be found in [19], [20]. The proposed control architecture for the output synchronization problem with event-triggered communication is inspired by the results reported in [5]. However, unlike [5], instead of requiring continuous communication links among controllers of different plants, here communication among different controllers is discrete in time, thereby making it more practical to implement in a real-world scenario. To derive the communication protocol that establishes when a broadcast should be carried out, we use as a starting point the results reported in [18] for event-triggered average consensus for plants with first and second order integrator dynamics. We proceed by extending these results for synchronization of generic LTI systems.

In summary, the contributions of this paper are:

1) the extension of the event-triggered consensus results in [18] for 1st and 2nd order integrators to event-triggered synchronization of LTI plants with arbitrary dynamics and directed communication links;

This research was partly supported by the FCT (PEst-OE/EEI/LA0009/2013). The work of J. Almeida was funded by grant SFRH/BD/30605/2006 from FCT.

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Version date: Oct 20th, 2014.

This article has been accepted for presentation at the 53rd IEEE Conference on Decision and Control, Los Angeles, CA, USA, December 2014.

2) a self-triggered implementation of the previous event-triggered synchronization protocol;
3) a solution for the output synchronization of heterogeneous LTI plants with event-triggered communication that guarantees bounded synchronization errors.

A. Notation

If $\{a_k\}_{k\geq 0}$ and $\{b_k\}_{k\geq 0}$ are two strictly increasing sequences of numbers in $\mathbb{R}$, then the union of both sequences is a strictly increasing sequence $\{c_k\}_{k\geq 0}$, where elements are reordered to satisfy the strictly increasing condition ($c_k < c_{k+1}$). We denote this by writing $\{c_k\}_{k\geq 0} = \{a_k\}_{k\geq 0} \cup \{b_k\}_{k\geq 0}$. For a signal $x : \mathbb{R}_+ \to \mathbb{R}^n$, if the limit from below at time $t \in \mathbb{R}_+$ exists it is defined as $x^-(t) = \lim_{s\uparrow t} x(s)$. If t is understood from context, we simply write x^- to stand for $x^-(t)$. A vector of dimension n whose entries are all equal to one is denoted by $\mathbf{1}_n$. For a matrix X, $\|X\|$ denotes its spectral norm defined as its largest singular value, and $\sigma(X)$ denotes its spectrum, that is, the set of eigenvalues of X.

B. Graph theory review

For an in-depth presentation of the subject, the reader is referred to, e.g., [21] for a comprehensive textbook on the matter and [22] for specific results regarding algebraic graph theory.

A directed graph or digraph $\mathcal{G} = \mathcal{G}(\mathcal{V}, \mathcal{E})$ consists of a finite set $\mathcal{V} = \{1, 2, \ldots, N\}$ of N vertices and a finite set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ of ordered pairs of vertices (i, j) named edges (in this paper, self-edges (i, i) are not allowed). If $(i, j) \in \mathcal{E}$, then we say that vertex i is an in-neighbor of vertex j and that j is an out-neighbor of vertex i. The set of in-neighbors and of out-neighbors of vertex i are defined as $\mathcal{N}_i^- = \{j \in \mathcal{V} : (j, i) \in \mathcal{E}\}$ and $\mathcal{N}_i^+ = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$, respectively. A directed path in $\mathcal{G}$ from vertex i to vertex j is a sequence of distinct edges of the form $\{(i, i_1), (i_1, i_2), \ldots, (i_k, j)\}$. A vertex i is a root of a graph $\mathcal{G}$ if there exists a path in $\mathcal{G}$ from vertex i to every other vertex in $\mathcal{G}$. If $\mathcal{G}$ has at least one root, we say that it is a rooted graph. The adjacency matrix of a digraph, denoted $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$, is a square matrix with rows and columns indexed by the vertices where $a_{ij} = 1$ if $(j, i) \in \mathcal{E}$ and is zero otherwise. The in-degree matrix $\mathcal{D}$ of a digraph is a diagonal matrix where the i, i-entry is equal to the in-degree of vertex i (cardinality of $\mathcal{N}_i^-$). The Laplacian of a digraph $\mathcal{L} = [l_{ij}] \in \mathbb{R}^{N\times N}$ is defined as $\mathcal{L} = \mathcal{D} - \mathcal{A}$. Next, we enumerate some important properties of the Laplacian.

Lemma 1: Let $\mathcal{L}$ denote the Laplacian of a graph $\mathcal{G}$. Then, the following properties hold:

1) $\mathcal{L}\mathbf{1}_N = 0$;
2) $\exists \beta \in \mathbb{R}^N, \beta^\top \mathbf{1}_N = 1 : \beta^\top \mathcal{L} = 0$;
3) $\sigma(\mathcal{L}) = \{0, \lambda_2, \ldots, \lambda_N\}$ with $\Re\{\lambda_i\} > 0$ for $i = 2, \ldots, N$;
4) $\mathcal{G}$ is a rooted graph if and only if 0 is a simple eigenvalue of $\mathcal{L}$;
5) if $\mathcal{G}$ is a rooted graph, then $\mathcal{L}$ may be written as
$$\mathcal{L} = \begin{bmatrix} \mathbf{1}_N & V \end{bmatrix} \operatorname{diag}(0, \mathcal{L}_{11}) \begin{bmatrix} \beta^\top \\ W \end{bmatrix}, \tag{1}$$
where $\sigma(\mathcal{L}_{11}) = \sigma(\mathcal{L})\backslash\{0\}$ and $V \in \mathbb{R}^{N\times(N-1)}$ and $W \in \mathbb{R}^{(N-1)\times N}$ are such that $\begin{bmatrix} \mathbf{1}_N & V \end{bmatrix}$ is nonsingular and $\begin{bmatrix} \beta^\top \\ W \end{bmatrix} = \left(\begin{bmatrix} \mathbf{1}_N & V \end{bmatrix}\right)^{-1}$ (in particular, this implies that $\mathbf{1}_N\beta^\top + VW = I_N$, $\beta^\top V = 0^\top$, $W\mathbf{1}_N = 0$, and $WV = I_{N-1}$).

II. Synchronization of LTI plants with event-triggered communication

Consider N heterogeneous LTI plants
$$\begin{aligned} \dot{x}_i &= A_i x_i + B_i u_i \\ y_i &= C_i x_i \\ z_i &= E_i x_i \end{aligned}$$
with state vector $x_i \in \mathbb{R}^{n_i}$, control input $u_i \in \mathbb{R}^{p_i}$, output vector $y_i \in \mathbb{R}^{q_i}$, and synchronization output $z_i \in \mathbb{R}^r$ for $i \in \{1, \ldots, N\}$. Matrices A_i, B_i, C_i, D_i, and E_i are of appropriate dimensions. The pair (A_i, B_i) is assumed controllable and the pairs (A_i, C_i) and (A_i, E_i) are assumed observable. Communication links established between different plants are represented by a fixed directed graph. The goal is to design decentralized control laws for u_i and a communication protocol that can guarantee that the synchronization errors $z_i(t) - z_j(t)$ stay bounded for all $i, j \in \{1, \ldots, N\}$ and $t \geq 0$.

In our proposed solution, we associate to each plant a local controller that consists of three components. The first one is the reference generator. Each reference generator has an internal state $\zeta_i \in \mathbb{R}^m$ that satisfies, for all $t \geq 0$,
$$\dot{\zeta}_i = S\zeta_i + v_i, \tag{2}$$
where v_i is the control input of the reference generator of plant i. The second component is a state observer whose internal state is represented by $\hat{x}_i$ that satisfies, for all $t \geq 0$,
$$\dot{\hat{x}}_i = A_i\hat{x}_i + B_i u_i + L_i(\hat{y}_i - y_i).$$
The third and final component is the control law that is defined as
$$u_i = K_i(\hat{x}_i - \Pi_i\zeta_i) + \Gamma_i\zeta_i,$$
where the matrices Π_i and Γ_i are obtained from the following assumption.

Assumption 1 ([5]): There exist a scalar $m \in \mathbb{N}$, $m \geq 2$ and matrices $S \in \mathbb{R}^{m\times m}$, $R \in \mathbb{R}^{q\times m}$, $\Pi_i \in \mathbb{R}^{n_i\times m}$, and $\Gamma_i \in \mathbb{R}^{p_i\times m}$ for $i \in \{1, \ldots, N\}$ such that $\sigma(S) \subset \bar{\mathbb{C}}_+$, the pair (S, R) is observable, and
$$\Pi_i S = A_i\Pi_i + B_i\Gamma_i \tag{3a}$$
$$R = E_i\Pi_i \tag{3b}$$
for $i \in \{1, \ldots, N\}$.

As we shall see, matrices S and R form a reference model that each plant should track. If we let
$$v_i = \sum_{j=1}^{N} a_{ij}(\zeta_j - \zeta_i) \tag{4}$$
where $a_{ij} = (\mathcal{A})_{ij}$ where $\mathcal{A}$ is the adjacency matrix of the communication graph, then the following is shown in [5].

Theorem 1: If the communication graph is rooted and K_i and L_i are such that $A_i + B_iK_i$ and $A_i + L_iC_i$ are Hurwitz

This article has been accepted for presentation at the 53rd IEEE Conference on Decision and Control, Los Angeles, CA, USA, December 2014.

for each $i \in \{1,\ldots,N\}$, then there exist $\delta \geq 1$ and $\lambda > 0$ such that

$$\|z_i(t) - Re^{St}z_0\| \leq \delta e^{-\lambda t}\|z_i(0) - Rz_0\|$$

for all $t \geq 0$ and all $i \in \{1,\ldots,N\}$, where $z_0 = (\beta^\top \otimes I_m)(\zeta_1(0),\ldots,\zeta_N(0))$.

In what follows, we modify (4) to avoid the need for continuous communication links among reference generators and propose an event-triggered communication protocol for its update. Let

$$v_i = \sum_{j=1}^{N} a_{ij}(\hat{\zeta}_j^i - \hat{\zeta}_i) \tag{5}$$

where $\hat{\zeta}_i$ and $\hat{\zeta}_j^i$ denote extra state variables of the reference generator that are updated depending on the source of the latest broadcast. The sequence of broadcast times of the reference generator associated with plant i is represented by $\{b_k^i\}_{k\geq 0}$ (where $b_0^i = 0$). The state variable $\hat{\zeta}_i$ evolves according to the reference model between broadcast times of plant i and is reset to the current value of ζ_i when a broadcast occurs. When a in-neighbor of plant i, say $j \in \mathcal{N}_i^-$, broadcasts a new value for ζ_j, then this value is used to reset the value of $\hat{\zeta}_j^i$. The dynamics of $\hat{\zeta}_i$ and $\hat{\zeta}_j^i$ may be written in the form of an impulsive system as

$$\begin{cases} \dot{\hat{\zeta}}_i = S\hat{\zeta}_i, & t \in [b_k^i, b_{k+1}^i), \quad (6a)\\ \hat{\zeta}_i = \zeta_i^-, & t = b_k^i, \quad (6b)\end{cases}$$

and

$$\begin{cases} \dot{\hat{\zeta}}_j^i = S\hat{\zeta}_j^i, & t \in [b_k^j, b_{k+1}^j), \quad (7a)\\ \hat{\zeta}_j^i = \zeta_j^-, & t = b_k^j, \quad (7b)\end{cases}$$

respectively. The sequence of broadcast times of plant i is generated according to

$$b_{k+1}^i = \inf\{t > b_k^i : |\hat{\zeta}_i(t) - \zeta_i(t)| = c(t)\}, \tag{8}$$

where $c(t)$ represents a time-varying threshold defined as

$$c(t) = c_0 + c_1 \mathrm{e}^{-\alpha t} \tag{9}$$

where $c_0, c_1, \alpha \geq 0$. The threshold starts at a value of $c_0 + c_1$ and then decreases monotonically, reaching c_0 asymptotically. This triggering scheme is inspired by the work reported in [18] for event-triggered consensus.

Without loss of generality, suppose that, for all $i, j \in \{1,\ldots,N\}$, $\hat{\zeta}_j^i$ is initialized with the value $\zeta_j(0)$. Since $\hat{\zeta}_j$ and $\hat{\zeta}_j^i$ have the same dynamics (cf. (6) with $i = j$ and (7), respectively), we have that $\hat{\zeta}_j^i(t) = \hat{\zeta}_j(t)$ for all $t \geq 0$. Therefore, for analysis purposes, only the state variable $\hat{\zeta}_j$ is required. However, if we allowed the broadcast data to arrive at each out-neighbor of plant i at different times due to, e.g., transmission delays, then the previous simplification would not be possible.

For the setup described above, we will show the following.

Theorem 2: If the communication graph is rooted and K_i and L_i are such that $A_i + B_iK_i$ and $A_i + L_iC_i$ are Hurwitz for each $i \in \{1,\ldots,N\}$, then there exist $c > 0$, such that

$$\|z_i(t) - Re^{St}z_0\| \leq c \tag{10}$$

for all $t \geq 0$ and all $i \in \{1,\ldots,N\}$.

Proof: The proof is given in Section II-C. ∎

A. Stability analysis of reference generators

Stacking the individual states ζ_i and $\hat{\zeta}_i$ into state vectors $\zeta = (\zeta_1,\ldots,\zeta_N)$ and $\hat{\zeta} = (\hat{\zeta}_1,\ldots,\hat{\zeta}_N)$, respectively, (2), (5), and (6) may be written in vector form as

$$\begin{cases} \begin{bmatrix}\dot{\zeta}\\ \dot{\hat{\zeta}}\end{bmatrix} = \begin{bmatrix} I_N \otimes S & -\mathcal{L}\otimes I_m \\ 0 & I_N \otimes S\end{bmatrix}\begin{bmatrix}\zeta\\ \hat{\zeta}\end{bmatrix}, t \in [b_k, b_{k+1}) \quad (11a)\\ \begin{bmatrix}\zeta\\ \hat{\zeta}\end{bmatrix} = \begin{bmatrix} I & 0 \\ R_k \otimes I_m & (I_N - R_k)\otimes I_m\end{bmatrix}\begin{bmatrix}\zeta^-\\ \hat{\zeta}^-\end{bmatrix}, t = b_k \quad (11b)\end{cases}$$

where $\{b_k\}_{k\geq 0} = \bigcup_{i=1}^N \{b_k^i\}_{k\geq 0}$ and $R_k = \operatorname{diag}(r_{1,k}, r_{2,k}, \ldots, r_{N,k})$ is a diagonal matrix whose entries satisfy

$$r_{i,k} = \begin{cases} 1, & \text{if } b_p^i = b_k \text{ for some } p \geq 0 \\ 0, & \text{otherwise}\end{cases}.$$

For analysis purposes, it is more convenient to work with the errors $e_i = \hat{\zeta}_i - \zeta_i$ that originate in the fact that $\hat{\zeta}_i$ is being used for feedback rather than ζ_i. The dynamics of e_i are given by

$$\begin{cases} \dot{e}_i = Se_i - \displaystyle\sum_{j=1}^{N} a_{ij}(\zeta_j + e_j - \zeta_i - e_i), t \in [b_k^i, b_{k+1}^i) \quad (12a)\\ e_i = 0, t = b_k^i. \quad (12b)\end{cases}$$

Using the error e_i, condition (8) may be rewritten as

$$b_{k+1}^i = \inf\{t > b_k^i : \|e_i(t)\| = c(t)\}. \tag{13}$$

In vector form, we have the dynamics

$$\begin{cases} \begin{bmatrix}\dot{\zeta}\\ \dot{e}\end{bmatrix} = \begin{bmatrix} Z & -\mathcal{L}\otimes I_m \\ \mathcal{L}\otimes I_m & I_N \otimes S + \mathcal{L}\otimes I_m\end{bmatrix}\begin{bmatrix}\zeta\\ e\end{bmatrix}, t \in [b_k, b_{k+1}) \quad (14a)\\ \begin{bmatrix}\zeta\\ e\end{bmatrix} = \begin{bmatrix} I & 0 \\ 0 & (I_N - R_k)\otimes I_m\end{bmatrix}\begin{bmatrix}\zeta^-\\ e^-\end{bmatrix}, t = b_k \quad (14b)\end{cases}$$

where $Z = I_N \otimes S - \mathcal{L}\otimes I_m$. The reference signal that each ζ_i should track is defined as

$$a(t) = (\beta^\top \otimes I_m)\zeta(t) \in \mathbb{R}^m \tag{15}$$

where β is defined in Lemma 1. Note that, if the graph is undirected, then L is symmetric and $a(t)$ becomes the average of all $\zeta_i(t)$. For all $t \in [b_k, b_{k+1})$, we have that

$$\begin{aligned}\dot{a} &= (\beta^\top \otimes I_m)\left(Z\zeta - (\mathcal{L}\otimes I_m)e\right)\\ &= (\beta^\top \otimes S)\zeta - ((\beta^\top\mathcal{L})\otimes I_m)(\zeta + e)\\ &= (1 \otimes S)(\beta^\top \otimes I_m)\zeta\\ &= Sa,\end{aligned}$$

where we have used the fact that $\beta^\top\mathcal{L} = 0$. When $t = b_k$, we have that $a = a^-$.

Let the disagreement vector be defined as $\delta(t) = \zeta(t) - \mathbf{1}_N \otimes a(t)$. It can be shown that δ satisfies, for all $t \geq 0$,

$$\begin{cases} \dot{\delta} = Z\delta - (\mathcal{L}\otimes I_m)e, & t \in [b_k, b_{k+1}), \quad (16a)\\ \delta = \delta^-, & t = b_k, \quad (16b)\end{cases}$$

This article has been accepted for presentation at the 53rd IEEE Conference on Decision and Control, Los Angeles, CA, USA, December 2014.

and $(\beta^\top \otimes I_m)\delta(t) = 0$. The norm of the disagreement vector is a measure of the level of synchrony between the state variables ζ_i. To derive a bound on the asymptotic behavior of δ, we need the following lemma.

Lemma 2: Let $v \in \mathbb{R}^{Nm}$ be such that $(\beta^\top \otimes I_m)v = 0$. If $\mathcal{G}$ is a rooted graph and, for all $i \in \{1, \ldots, m\}$ and $j \in \{1, \ldots, N-1\}$,

$$\Re\{\lambda_i(S) - \lambda_j(\mathcal{L}_{11})\} < 0, \tag{17}$$

then there exist $\kappa \geq 1$ and $\lambda > 0$ such that, for all $t \geq 0$,

$$\|\mathsf{e}^{Zt}v\| \leq \kappa \mathsf{e}^{-\lambda t}\|v\|. \tag{18}$$

Proof: Let $\mathcal{L}$ be decomposed as in (1). Then, the matrix Z may be written as

$$Z = \bar{V}\left(I_N \otimes S - \operatorname{diag}(0, \mathcal{L}_{11}) \otimes I_m\right)\bar{W},$$

where $\bar{V} = \left(\begin{bmatrix}\mathbf{1}_N & V\end{bmatrix} \otimes I_m\right)$ and $\bar{W} = \begin{bmatrix}\beta^\top \\ W\end{bmatrix} \otimes I_m = \left(\begin{bmatrix}\mathbf{1}_N & V\end{bmatrix} \otimes I_m\right)^{-1}$. Then,

$$\mathsf{e}^{Zt}v = \bar{V}\mathsf{e}^{(I_N \otimes S - \operatorname{diag}(0,\mathcal{L}_{11}) \otimes I_m)t}\bar{W}v.$$

Since $(\beta^\top \otimes I_m)v = 0$, we have that

$$\mathsf{e}^{Zt}v = (V \otimes I_m)\,\mathsf{e}^{(I_{N-1} \otimes S - \mathcal{L}_{11} \otimes I_m)t}\,(W \otimes I_m)\,v.$$

Note that the eigenvalues of $I_{N-1} \otimes S - \mathcal{L}_{11} \otimes I_m$ are of the form $\lambda_i(S) - \lambda_j(\mathcal{L}_{11})$. Therefore, it is Hurwitz by hypothesis. Hence, there exist $\kappa_1 \geq 1$ and $\lambda > 0$ such that, for all $t \geq 0$,

$$\|\mathsf{e}^{(I_{N-1} \otimes S - \mathcal{L}_{11} \otimes I_m)t}\| \leq \kappa_1 \mathsf{e}^{-\lambda t},$$

from which we conclude that

$$\|\mathsf{e}^{Zt}v\| \leq \underbrace{\kappa_1\|V\|\|W\|}_{\kappa}\mathsf{e}^{-\lambda t}\|v\|, \tag{19}$$

where we used the fact that $\|X \otimes I\| = \|X\|$. ■

Lemma 2.1 in [18] is recovered by taking $\mathcal{G}$ undirected connected graph ($\mathcal{L}$ symmetric) and S equal to the zero matrix in Lemma 2 (in which case $\kappa = 1$ and $\lambda = \lambda_2(\mathcal{L})$). Note that if all eigenvalues of S are imaginary, that is, $\sigma(S) \subset \mathrm{i}\mathbb{R}$, then Lemma 2 holds for any rooted graph. Using Lemma 2 in (16), we prove the following.

Theorem 3: Assume Zeno behavior is avoided. If $\lambda > 0$, then, for all initial conditions $\zeta(0) \in \mathbb{R}^{Nm}$ and all $0 < \alpha < \lambda$, the disagreement vector δ satisfies

$$\|\delta(t)\| \leq \bar{\delta} = \kappa \max\left\{\|\delta(0)\|, \sqrt{N}\|\mathcal{L}\|\left(\frac{c_0}{\lambda} + \frac{c_1}{\lambda - \alpha}\right)\right\}, \tag{20}$$

for all $t \geq 0$, and

$$\lim_{t \to +\infty}\|\delta(t)\| \leq \bar{\delta}_\infty = \frac{\kappa\sqrt{N}\|\mathcal{L}\|}{\lambda}c_0. \tag{21}$$

Proof: The proof follows along the same lines of Theorem 3.2 in [18]. ■

To show that the closed-loop system does not exhibit Zeno behavior, we will prove that the individual sequences $\{b_k^i\}_{k \geq 0}$ have a minimum time between broadcasts. This implies that the sequence $\{b_k\}_{k \geq 0}$ cannot have any accumulation points since the intersection of any closed interval with $\{b_k\}_{k \geq 0}$ can only have a finite number of elements.

Lemma 3: If $c_0 > 0$, then there exists $\theta_{\min} > 0$ such that, for all $k \geq 0$ and all $i = 1, \ldots, N$, $b_{k+1}^i - b_k^i \geq \theta_{\min}$.

Proof: Let $k \geq 0$ be fixed and $i \in \{1, \ldots, N\}$ be given. Consider the dynamics of e_i given in (12). Solving for $t \geq b_k^i$ and using the fact that $e_i(b_k^i) = 0$ yields

$$e_i(t) = -\int_{b_k^i}^{t}\mathsf{e}^{S(t-s)}v_i(s)\mathrm{d}s$$

for all $t \in [b_k^i, b_{k+1}^i)$. Then,

$$\|e_i(t)\| \leq \int_{b_k^i}^{t}\|\mathsf{e}^{S(t-s)}\|\|v_i(s)\|\mathrm{d}s \leq \int_{b_k^i}^{t}\mathsf{e}^{\omega(t-s)}\|v_i(s)\|\mathrm{d}s \tag{22}$$

where we used the fact that there exists $\omega > 0$ such that $\|\mathsf{e}^{St}\| \leq \mathsf{e}^{\omega t}$ for all $t \geq 0$. Using (20), we have that

$$\begin{aligned}\|v_i\| \leq \|v\| &= \|(\mathcal{L} \otimes I_m)(\zeta + e)\| \\ &= \|(\mathcal{L} \otimes I_m)(\delta + e)\| \\ &\leq \|\mathcal{L}\|(\|\delta\| + \|e\|) \\ &\leq \|\mathcal{L}\|(\bar{\delta} + \sqrt{N}(c_0 + c_1)) = \bar{v}.\end{aligned} \tag{23}$$

Replacing (23) in (22), yields

$$\|e_i(t)\| \leq \int_{b_k^i}^{t}\mathsf{e}^{\omega(t-s)}\bar{v}\mathrm{d}s = \frac{\bar{v}}{\omega}\left(\mathsf{e}^{\omega(t-b_k^i)} - 1\right).$$

Hence, a lower bound on the minimum time interval between two consecutive broadcast times of plant i is given by

$$\frac{\bar{v}}{\omega}\left(\mathsf{e}^{\omega\theta_{\min}} - 1\right) = c_0 \Leftrightarrow \theta_{\min} = \frac{1}{\omega}\log\left(1 + \frac{\omega c_0}{\bar{v}}\right) > 0. \tag{24}$$

Since this bound is independent of both k and i, it holds for all $k \geq 0$ and $i \in \{1, \ldots, N\}$. ■

The results obtained in [18], namely Lemma 2.1 and Theorem 3.2, are recovered by taking $S = 0$ and considering only undirected connected graphs in Lemma 2, Theorem 3, and Lemma 3.

B. Self-triggered communication

To avoid spending computational resources constantly testing if the broadcast condition has been violated, in this section we propose a self-triggered implementation of the event-triggered communication protocol defined in (13).

Consider the dynamics associated with the error e_i given in (12) that may be written as

$$\dot{e}_i = Se_i + \sum_{j=1}^{N}a_{ij}(\hat{\zeta}_j - \hat{\zeta}_i) = Se_i + v_i. \tag{25}$$

The self-triggered communication protocol proceeds as follows. Suppose the reference generator of plant i executes a broadcast of $\hat{\zeta}_i$ at time $t = b_k$ (that is, $b_k = b_p^i$ for some $p \in \mathbb{Z}_+$). At this point, instead of continuously testing the event condition defined in (13) to obtain the next broadcast time b_{k+1}^i, plant i computes b_{k+1}^i using the information available at the current time instant b_k. At the same time, all its out-neighbors have to recompute their next broadcast time b_{k+1}^j as well to guarantee that their corresponding event conditions are satisfied. This is necessary because when $\hat{\zeta}_i$

This article has been accepted for presentation at the 53rd IEEE Conference on Decision and Control, Los Angeles, CA, USA, December 2014.

is updated, $v_j(b_k)$ changes for all $j \in \mathcal{N}_i^+$ thereby altering the evolution of ζ_j and e_j for $t \geq b_k$.

In what follows, we consider $j \in \{i\} \cup \mathcal{N}_i^+$. To derive an expression for the computation of the broadcast times, we start by solving (25) in t, yielding

$$e_j(t) = \mathsf{e}^{S(t-b_k)} e_j(b_k) - \int_{b_k}^{t} \mathsf{e}^{S(t-s)} v_j(s) \mathrm{d}s, \tag{26}$$

for all $t \geq b_k$. Notice that $e_i(b_k) = e_i(b_k^i) = 0$ but, in general, $e_j(b_k) \neq 0$ for $j \in \mathcal{N}_i^+$. Given (26), finding a closed-form solution for the equation $\|e_j(t)\| = c(t)$ is, in general, impossible. Instead of the exact solution $b_{k+1}^{j,*}$, we will find an approximate solution b_{k+1}^j such that $b_{k+1}^j \leq b_{k+1}^{j,*}$ thereby guaranteeing that the event condition in (13) is satisfied. The goal is to keep the gap between b_{k+1}^j and $b_{k+1}^{j,*}$ as small as possible. The self-triggered implementation is therefore expected to generate a sequence of broadcast times with an higher broadcast rate than the one obtained with the event-triggered implementation.

Note that the dynamics of $\hat{\zeta}_i$ given in (6) imply that, for $t \in [b_k, b_{k+1})$,

$$\hat{\zeta}_i(t) = \mathsf{e}^{S(t-b_k)} \zeta_i(b_k).$$

Thus, v_i in (25) may be written as

$$v_i(t) = \sum_{j=1}^{N} a_{ij}(\hat{\zeta}_j - \hat{\zeta}_i) = \mathsf{e}^{S(t-b_k)} \bar{v}_i(b_k).$$

where $\bar{v}_i(b_k) = \sum_{j=1}^{N} a_{ij}(\hat{\zeta}_j(b_k) - \hat{\zeta}_i(b_k))$. Solving (25) in t yields

$$\begin{aligned} e_j(t) &= \mathsf{e}^{S(t-b_k)} e_j(b_k) - \int_{b_k}^{t} \mathsf{e}^{S(t-s)} \mathsf{e}^{S(s-b_k)} \bar{v}_j(b_k) \mathrm{d}s \\ &= \mathsf{e}^{S(t-b_k)} e_j(b_k) - (t-b_k) \mathsf{e}^{S(t-b_k)} \bar{v}_j(b_k), \end{aligned}$$

from which we obtain

$$\|e_j(t)\| \leq \mathsf{e}^{\omega(t-b_k)} \left(\|e_j(b_k)\| + \|\bar{v}_j(b_k)\| (t-b_k) \right).$$

Then, the next broadcast time is defined as $b_{k+1}^j = b_k + \theta_j^*$ where θ_j^* is the positive solution of

$$\begin{aligned} &\mathsf{e}^{\omega \theta_j^*} \left(\|e_j(b_k)\| + \|\bar{v}_j(b_k)\| \theta_j^* \right) = c(b_k + \theta_j^*) \\ \Leftrightarrow &\|e_j(b_k)\| + \|\bar{v}_j(b_k)\| \theta_j^* = c_0 \mathsf{e}^{-\omega \theta_j^*} + c_1 \mathsf{e}^{-\alpha b_k} \mathsf{e}^{-(\omega+\alpha)\theta_j^*}. \end{aligned} \tag{27}$$

Note that $e_j(b_k)$ and $\bar{v}_j(b_k)$ are known to plant j at time $t = b_k$, thus they may be used to compute the next broadcast time. Although, in general, there is no closed-form solution for (27), it can be efficiently solved numerically. Taking $c_1 = \alpha = \|e_j(b_k)\| = 0$ and using the bound in (23), a lower bound on the minimum broadcast interval of all plants is denoted by $\theta_{\min}^{\text{self}}$ and is defined as the positive solution of $\bar{v}\theta = c_0 \mathsf{e}^{-\omega\theta}$. It can be shown that $\theta_{\min}^{\text{self}} \leq \theta_{\min}$ holds.

C. Proof of Theorem 2

Let $\tilde{x}_i = x_i - \hat{x}_i$ and $\epsilon_i = x_i - \Pi_i \zeta_i$. The dynamics of $\tilde{x}_i$ is given by

$$\dot{\tilde{x}}_i = (A_i + L_i C_i) \tilde{x}_i.$$

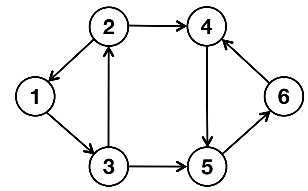


Fig. 1. Communication graph of example.

Since the matrix $A_i + L_i C_i$ is Hurwitz, we have that $\tilde{x}_i \to 0$. The dynamics of ϵ_i is given by

$$\begin{aligned} \dot{\epsilon}_i &= A_i x_i + B_i u_i - \Pi_i (S_i \zeta_i + v_i) \\ &= A_i x_i + B_i \left(K_i(\epsilon_i - \tilde{x}_i) + \Gamma_i \zeta_i \right) - \Pi_i (S_i \zeta_i + v_i) \\ &\overset{\text{(3a)}}{=} (A_i + B_i K_i) \epsilon_i - B_i K_i \tilde{x}_i - \Pi_i v_i. \end{aligned}$$

In the proof of Lemma 3, we have shown that v_i is bounded. Since the matrix $A_i + B_i K_i$ is Hurwitz, ϵ_i is input-to-state stable with respect to v_i which implies that ϵ_i is bounded as well. Finally, note that a defined in (15) satisfies $a(t) = \mathsf{e}^{St}(\beta^\top \otimes I_m)\zeta(0)$ for all $t \geq 0$. Therefore, we have that

$$\begin{aligned} z_i - Ra &= E_i \left(\epsilon_i + \Pi_i \zeta_i \right) - E_r a \\ &= E_i \epsilon_i + R \left(\zeta_i - a \right). \end{aligned}$$

Since the term $E_i \epsilon_i$ is bounded from the above analysis and the term $R(\zeta_i - a)$ is bounded by Theorem 3, we conclude that (10) holds.

III. Example

To illustrate the proposed control architecture, we consider $N = 6$ different plants with plant matrices given in Table I. The plants are required to synchronize to the dynamics of an harmonic oscillator, that is,

$$S = \begin{bmatrix} 0 & -1 \\ 1 & 0 \end{bmatrix}, R = \begin{bmatrix} 1 & 0 \end{bmatrix},$$

using both event and self-triggered communication protocols. For each plant, the controller gains K_i are obtained by solving an LQR control problem with weight matrices $Q_{\text{LQR}} = I_{n_i}$ and $R_{\text{LQR}} = I_{p_i}$. For the observer gain L_i, we use the gain of a steady-state Kalman filter with noise covariances $Q_{\text{KF}} = I_{n_i}$ and $R_{\text{KF}} = I_{q_i}$. Matrices Π_i and Γ_i are computed by solving (3). The communication graph is shown in Fig. 1. The triggering parameters are $c_0 = 0.001$, $c_1 = 0.499$, and $\alpha = \frac{1}{4}$. The initial conditions of the reference generators are $\zeta(0) = \hat{\zeta}(0) = \begin{bmatrix} 0.8 & 0 & 0.5 & 0.8 & -0.5 & 0.9 & -1.1 & 0 & -0.6 & -1 & 0.7 & -1.1 \end{bmatrix}$. The initial conditions of the plants are $x(0) = \begin{bmatrix} 0.1 & -0.2 & 0 & 0.3 & 0 & 0.1 & 0 & -0.2 & 0 & 0.3 & 0 & 0.1 & -0.3 & -0.2 & 0 & 0 & 0 & 0.2 & 0 & -0.1 & 0 \end{bmatrix}$. The observers are initialized with $\hat{x}_i(0) = 0$ for all i.

The simulation results obtained are shown in Fig. 2. It can be seen that the synchronization output z_i of every plant converges to a neighborhood of the desired reference signal. From the evolution of the synchronization error, we see that the performance of both protocols is very similar although the self-triggered approach requires a slightly higher number of broadcasts. For example, plant 6 made 34 and 35 broadcasts using the event-triggered and the self-triggered communication protocols, respectively.

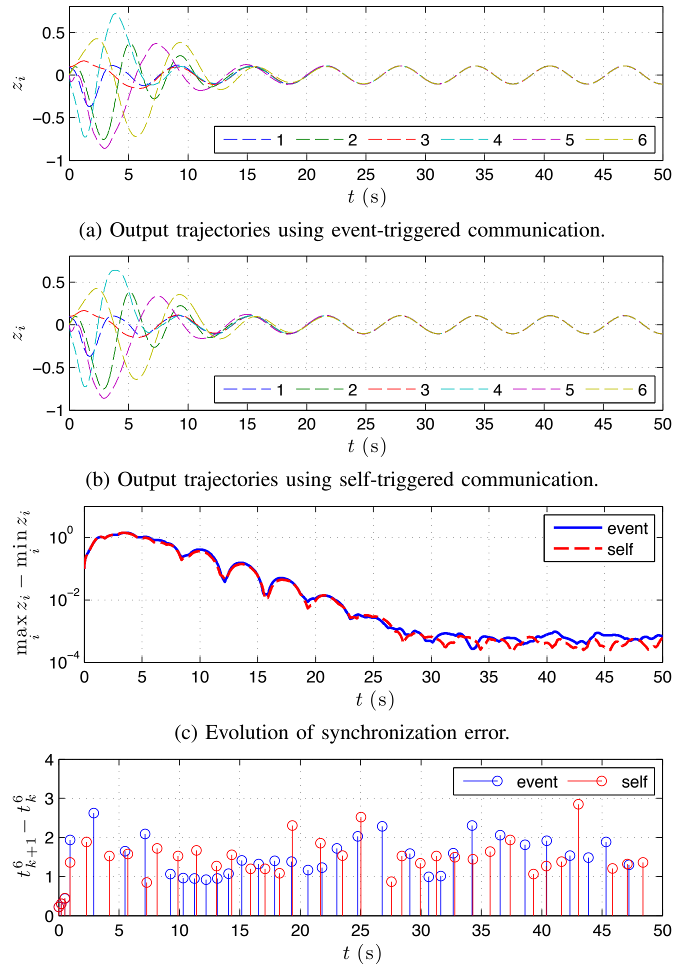
This article has been accepted for presentation at the 53rd IEEE Conference on Decision and Control, Los Angeles, CA, USA, December 2014.

TABLE I
PLANT MATRICES OF EXAMPLE.

i	A_i	B_i	C_i	E_i
1	$\begin{bmatrix} 0 & 1 \\ -2 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \end{bmatrix}$
2	$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 3 & -2 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$
3	$\begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -1 & 2 & -3 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 \end{bmatrix}$
4	$\begin{bmatrix} 1 & 1 & 0 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 1 & 0 \\ 1 & 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 \\ 0 & 0 \\ 0 & 1 \\ 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 & 0 \end{bmatrix}$
5	$\begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 0 \\ 0 & 0 & 0 & 1 \\ \frac{1}{2} & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 \\ 1 \\ 0 \\ 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 1 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 1 & 1 \end{bmatrix}$
6	$\begin{bmatrix} 0 & 1 & 1 & 0 & 0 \\ -1 & 0 & 1 & 0 & 0 \\ 0 & 1 & 0 & -1 & 0 \\ 1 & 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 \\ 0 & 0 \\ 0 & 0 \\ 1 & 0 \\ 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \end{bmatrix}$	$\begin{bmatrix} 1 & 0 & 0 & 0 & 0 \end{bmatrix}$

IV. CONCLUSION

This paper proposed a control architecture to achieve output synchronization of a group of heterogeneous LTI plants. Results from [5] and [18] where employed to arrive at controllers capable of approximately synchronizing the plants while communicating only sporadically using either event-triggered or self-triggered communication protocols. Ongoing work is been conducted to include event-triggered control mechanisms in the local tracking controller of each plant, thereby arriving at a control architecture that employs event-triggered techniques in every control component.



(a) Output trajectories using event-triggered communication.

(b) Output trajectories using self-triggered communication.

(c) Evolution of synchronization error.

(d) Sequence of broadcast times of plant 6.

Fig. 2. Simulation results of the example.

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