Synchronization of multi-agent systems using event-triggered and self-triggered broadcasts

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Abstract—This paper addresses the problem of synchronizing a group of identical linear time-invariant agents that exchange information through a communication network. The agents may only broadcast information at discrete time instants and the decision to execute a broadcast is based on an event-triggered communication protocol. We prove that with the proposed control architecture the state of each agent converges to and remains in a neighborhood of a desired reference signal and the closed-loop system does not exhibit Zeno solutions. A self-triggered implementation of the proposed event-triggered communication protocol is also derived.

I. INTRODUCTION

In this paper, we define a multi-agent system as a dynamical system formed by a set of agents, each with dynamics modeled by a linear time-invariant (LTI) system, connected by a communication network that provides them with the means to exchange information. A survey of applications of multi-agent systems presented in [1] illustrates how local decentralized coordination strategies can be employed so that a desired global behavior is observed. A special class of applications requires the agents to align their states in a well-defined sense, with the most representative examples being the consensus and synchronization problems (see, e.g., [1]–[5]).

We address the synchronization problem for groups of identical agents. Although the authors of [5] solve this problem for groups of heterogeneous agents, our goal is to drop the assumption of continuous communication links present in [5] by employing sampled-data control techniques. The objective is to derive decentralized control laws and communication protocols capable of making the state of each agent converge to the same reference signal.

Due to the digital nature of the communication network, an additional constraint on the protocol design arises from the fact that communications can only occur at discrete time instants. The standard approach would be to broadcast information periodically. However, in recent years a different strategy has received attention due to a flurry of theoretical developments. Known as event-triggered control, in this new approach such tasks as sampling a signal or broadcasting information are only executed when deemed necessary according to some triggering conditions, often dependent on the state of each agent. For more details on this approach, the interested reader is referred to, e.g., [6]–[9] for the single plant case and to [10]–[12] for the case of multiple plants. It is important to point out that in event-triggered control, triggering conditions must be constantly monitored which may be infeasible for some applications. To circumvent this issue, self-triggered control strategies were developed where instead of continuously testing a triggering condition, an event scheduler computes when the next event should occur by using information available at the current time instant (see, e.g., [13]–[16]).

In a multi-agent scenario where agents have to communicate with each other, the event-triggered strategy is even more relevant since the communication medium is often shared by all agents, meaning that if each agent tried to transmit too often, successful communications would become impossible. Hence, by resorting to event-triggered control techniques, a communication protocol that avoids redundant broadcasts of information is sought. These techniques have been applied to the consensus problem in [17]–[20]. We note that the consensus problem is a particular type of synchronization problem where the reference signal is constant.

The contribution of this paper is twofold: i) we extend the event-triggered consensus results reported in [18] for 1st and 2nd order integrators with an undirected communication network; this is done by deriving an event-triggered communication protocol capable of achieving synchronization for a class of agents with LTI dynamics that are connected by a directed communication network and ii) we offer a self-triggered implementation of the proposed event-triggered communication protocol.

Notation: If \( \{a_k\}_{k \geq 0} \) and \( \{b_k\}_{k \geq 0} \) are two strictly increasing sequences with elements in \( \mathbb{R} \), then their union is a sequence \( \{c_k\}_{k \geq 0} \) defined as the set of unique elements in \( \{a_k\}_{k \geq 0} \) and \( \{b_k\}_{k \geq 0} \) reordered to satisfy \( c_k < c_{k+1} \) for all \( k \geq 0 \). We denote this by writing \( \{c_k\}_{k \geq 0} = \{a_k\}_{k \geq 0} \cup \{b_k\}_{k \geq 0} \). For a complex number \( z \), \( \mathbb{R}\{z\} \) denotes its real part.

For a signal \( x : [0, +\infty) \rightarrow \mathbb{R}^n \), if the limit from below at time \( t \in [0, +\infty) \) exists, then it is defined as \( x^{-}(t) = \lim_{s \uparrow t} x(s) \). If \( t \) is understood from context, we simply write \( x \) and \( x^{-} \) to stand for \( x(t) \) and \( x^{-}(t) \), respectively. A vector of dimension \( n \) whose entries are all equal to one is denoted by \( 1_n \). Given a collection of vectors \( \{x_1, \ldots, x_N\} \) where \( x_i \in \mathbb{R}^{n_i} \), the vector obtained by stacking all \( x_i \) column-wise is represented by \( z = (x_1, \ldots, x_N) = \begin{bmatrix} x_1^T & \ldots & x_N^T \end{bmatrix}^T \). The symbol \( I_n \)
denotes the identity matrix of dimension \( n \). For a square matrix \( X \), \( e^X \), \( \|X\| \), and \( \sigma(X) \) denote its matrix exponential, its spectral norm (defined as its largest singular value), and its spectrum (the set of eigenvalues of \( X \)), respectively. The symbol \( \otimes \) denotes the Kronecker product.

II. GRAPH THEORY REVIEW

In this section we introduce some necessary concepts and results from graph theory (adapted from [1], [21]) required for the presentation and analysis of our proposed solution for the problem of event-triggered synchronization.

A (directed) graph \( \mathcal{G} = (\mathcal{V}, \mathcal{E}) \) consists of a finite set \( \mathcal{V} = \{1, 2, \ldots, N\} \) of vertices and a finite set \( \mathcal{E} \subseteq \mathcal{V} \times \mathcal{V} \) of \( m \) ordered pairs of vertices \((i, j)\) named edges (in this paper, self-edges \((i, i)\) are not allowed). An undirected graph is defined as a graph where \((i, j) \in \mathcal{E}\) if and only if \((j, i) \in \mathcal{E}\). If \((i, j) \in \mathcal{E}\), then we say that vertex \( i \) is an in-neighbor of vertex \( j \) and that \( j \) is an out-neighbor of vertex \( i \). The set of in-neighbors and the set of out-neighbors of vertex \( i \) are defined as \( \mathcal{N}^-_i = \{ j \in \mathcal{V} : (j, i) \in \mathcal{E} \} \) and \( \mathcal{N}^+_i = \{ j \in \mathcal{V} : (i, j) \in \mathcal{E} \} \), respectively. A path in \( \mathcal{G} \) from vertex \( i \) to vertex \( j \) is a sequence of distinct edges of the form \((i, i_1), (i_1, i_2), \ldots, (i_k, j)\). A vertex \( i \) is a root of a graph \( \mathcal{G} \) if there exists a path in \( \mathcal{G} \) from vertex \( i \) to every other vertex in \( \mathcal{G} \). If \( \mathcal{G} \) has at least one root, we say that it is a rooted graph. If a graph \( \mathcal{G} \) is undirected and rooted, then it is said to be connected (in this case, all vertices are roots). A weighted graph is a graph where a real number (weight) is associated with every edge in the graph (in this paper, all graphs are weighted). The adjacency matrix of a graph, denoted by \( A = [a_{ij}] \in \mathbb{R}^{N \times N} \), is a square matrix with rows and columns indexed by the vertices and whose entries satisfy \( a_{ij} = 1 \) if \((i, j) \in \mathcal{E}\) and zero otherwise (\( a_{ij} \) denotes the weight for edge \((j, i) \in \mathcal{E}\)). The in-degree matrix \( D \) of a graph is a diagonal matrix where the \((i, i)\)-entry is equal to the in-degree of vertex \( i \) defined as \( \sum_{j=1}^{N} a_{ij} \). The Laplacian matrix of a graph \( \mathcal{G} \) is defined as \( \mathcal{L} = D - A \) and has the following properties: i) \( \mathcal{L} \mathbf{1}_N = 0 \) and there exists \( \beta \in \mathbb{R}^N, \beta^T \mathbf{1}_N = 1 \) such that \( \beta^T \mathcal{L} = 0 \); ii) \( \sigma(\mathcal{L}) = \{\lambda_1, \lambda_2, \ldots, \lambda_N\} \) with \( \mathbb{R}\{\lambda_1\} > 0 \) for all non-zero eigenvalues; iii) \( \mathcal{G} \) is a rooted graph if and only if 0 is a simple eigenvalue of \( \mathcal{L} \); iv) if \( \mathcal{G} \) is a rooted graph, then there exist matrices \( \mathcal{E} \in \mathbb{R}^{(N-1)\times(N-1)}, \mathcal{V} \in \mathbb{R}^{N\times(N-1)}, \) and \( \mathcal{W} \in \mathbb{R}^{(N-1)\times N} \) such that \( \sigma(\mathcal{L}) = \sigma(\mathcal{L}) \setminus \{0\} \), \( \mathcal{L} = \begin{bmatrix} \beta & \mathcal{W}^T \\ \mathcal{V}^T & 0 \end{bmatrix} \mathcal{L}^{(0, \mathcal{L})} \begin{bmatrix} \beta & \mathcal{W}^T \end{bmatrix}^T \) is nonsingular, \( \begin{bmatrix} \beta & \mathcal{W}^T \end{bmatrix}^T \begin{bmatrix} 1_N & \mathcal{V} \end{bmatrix} = 1 \), and \( \mathcal{L} = \begin{bmatrix} 1_N & \mathcal{V} \end{bmatrix} \mathcal{L}^{(0, \mathcal{L})} \begin{bmatrix} \beta & \mathcal{W}^T \end{bmatrix}^T \).

III. SYNCHRONIZATION OF MULTI-AGENT SYSTEMS

The multi-agent system that we consider consists of \( N \) agents with identical LTI dynamics. Each agent has a state denoted by \( \zeta_i \in \mathbb{R}^m \) such that \( \zeta_i(0) \in \mathbb{R}^m \) and, for all \( t \geq 0 \),

\[
\dot{\zeta}_i = A_r \zeta_i + v_i
\]

(2)

where \( v_i \in \mathbb{R}^m \) is the control input and \( A_r \in \mathbb{R}^{m\times m} \) (\( A_r \) may have unstable eigenvalues). To achieve synchronization, the state \( \zeta_i \) must evolve in such a way that its trajectory is eventually the same across all agents. Note that due to different initial conditions, the agents are not guaranteed to converge to the same trajectory. In order to correct this misalignment, the agents must exchange information among them by using a communication network. This network is modeled by a communication graph \( \mathcal{G} \) that is assumed to be fixed over time and where an edge \((i, j) \in \mathcal{E}\) means that agent \( i \) receives information from agent \( j \). Convergence to the same trajectory must be distributed in the sense that \( v_i \) may only depend on information obtained from the in-neighbors of agent \( i \). When continuous communication links among agents are allowed, one possible solution is to let \( v_i \) be given by

\[
v_i = \sum_{j=1}^{N} a_{ij}(\zeta_j - \zeta_i),
\]

(3)

where \( a_{ij} \) denotes the entries of the adjacency matrix associated with \( \mathcal{G} \). In [3] it is shown that if the following assumption is satisfied, then all agents synchronize asymptotically, that is, for all initial conditions \( \zeta_i(0) \in \mathbb{R}^m, \lim_{t \to +\infty} \|\zeta_i(t) - \zeta_j(t)\| = 0 \), for all \( i, j \in \{1, \ldots, N\} \).

Assumption 1. The communication graph \( \mathcal{G} \) is a rooted graph and, for all \( i \in \{1, \ldots, m\} \) and \( j \in \{2, \ldots, N\} \),

\[
\mathbb{R}\{\lambda_i(\mathcal{L}_r) - \lambda_j(\mathcal{L}_r)\} < 0.
\]

(4)

Assumption 1 requires the connectivity of the graph to be strong enough to dominate the unstable dynamics in \( A_r \), so that (4) holds. One way to accomplish this is to design the graph topology or edge weights such that the non-zero eigenvalues of \( \mathcal{L} \) meet condition (4). This is always possible, as shown next. Suppose \( \mathcal{G}_p \) is a rooted graph with all edge weights equal to \( \rho > 0 \) and let \( \mathcal{L}_p \) denote its corresponding Laplacian matrix. Notice that \( \rho \mathcal{L}_p = \rho \mathcal{L}_1 \). In this case, the stability condition (4) becomes

\[
\rho > \max_{i,j=1}^{m} \frac{\mathbb{R}\{\lambda_i(\mathcal{L}_r)\}}{\min_{i,j=1}^{m} \mathbb{R}\{\lambda_j(\mathcal{L}_1)\}} = \max_i \mathbb{R}\{\lambda_i(\mathcal{L}_r)\}.
\]

(5)

Thus, by selecting \( \rho \) sufficiently large, the unstable dynamics of \( A_r \) (eigenvalues of \( A_r \) with positive real part) can be dominated. Note that if all eigenvalues of \( A_r \) are imaginary, that is, \( \sigma(A_r) \subset i\mathbb{R} \), then (4) is satisfied for all rooted graphs.

A. Event-triggered synchronization

To avoid the need for continuous communication links in (3) and inspired by the work reported in [18] for event-triggered consensus, in this section we propose an event-triggered solution for the multi-agent synchronization problem. The proposed control architecture is represented in Fig. 1, from the point of view of agent \( i \). The agent has been augmented with additional state variables and is responsible for deciding when its current state should broadcast to the network, as represented by the broadcast event detector. This event detector triggers the broadcast of the current value of \( \zeta_i \) to the out-neighbors of agent \( i \) whenever a given state dependent condition is violated. The sequence of time instants where these violations occur is referred to as the sequence of broadcast times of agent \( i \) and is denoted by \( \{b_i^k\}_{k \geq 0} \).
The setup described in [18] for event-triggered consensus is recovered by taking \( m = 1 \) and \( A_r = 0 \). In this case, the authors proved the following result.

**Theorem 1** ([18, Theorem 3.2]). If \( G \) is an undirected connected graph and \( c_0 > 0 \), then the closed-loop system does not exhibit Zeno solutions and each agent’s trajectory satisfies

\[
\lim_{t \to +\infty} \| \zeta_i(t) - a \| \leq \frac{\sqrt{N} \| L \|}{\lambda_2(L)} c_0,
\]

for all \( \zeta_i(0) \in \mathbb{R}^m \), where \( a = \frac{1}{N} \sum_{i=1}^{N} \zeta_i(0) \) and \( \lambda_2(L) \) is the smallest nonzero eigenvalue of \( L \).

In the next section, we extend Theorem 1 by allowing directed graphs and an arbitrary \( A_r \) matrix as long as Assumption 1 is satisfied.

**B. Stability analysis**

For analysis purposes, it is more convenient to work with the errors \( e_i = \hat{\zeta}_i - \zeta_i \) that originate from the fact that \( \hat{\zeta}_i \) is used for feedback rather than \( \zeta_i \). The dynamics of \( e_i \) are given by

\[
\begin{align*}
\dot{e}_i &= A_v e_i - v_i, & t &\in [b_k^i, b_{k+1}^i), \\
e_i(0) &= 0, & t &= b_k^i.
\end{align*}
\]

Using the error \( e_i \), (10) is equivalent to

\[
b_{k+1}^i = \inf \{ t > b_k^i : \| e_i(t) \| = c(t) \}.
\]

Let \( \zeta = (\zeta_1, \ldots, \zeta_N) \) and \( e = (e_1, \ldots, e_N) \) denote new state vectors. Their dynamics are derived from (2), (9), and (12) and may be written as

\[
\begin{align*}
\dot{\zeta} &= L \otimes I_m e, & t &\in [b_k^i, b_{k+1}^i), \\
\zeta(0) &= (I_N \otimes I_m) e, & t &= b_k^i,
\end{align*}
\]

where \( \{ b_k \}_{k \geq 0} = \bigcup_{i=1}^{N} \{ b_k^i \}_{k \geq 0} \). \( Z = I_N \otimes A_r - L \otimes I_m \), and \( R_k = \text{diag}(r_{1,k}, r_{2,k}, \ldots, r_{N,k}) \) is a diagonal matrix whose entries satisfy \( r_{i,k} = 1 \) if \( b_P^i = b_k^i \) for some \( p \geq 0 \) and are zero otherwise.

We will show that each \( \zeta_i \) converges to a neighborhood of the reference signal \( a(t) = (\beta^T \otimes I_m)\zeta(t) \). Note that if the graph is undirected, then \( L \) is symmetric, \( \beta = 1_N / N \), and \( a(t) \) becomes the average of all \( \zeta_i(t) \). The signal \( a \) satisfies

\[
\begin{align*}
\dot{a} &= (\beta^T \otimes I_m)(Z \zeta - (L \otimes I_m)e) \\
&= (\beta^T \otimes A_r)\zeta - ((\beta^T \otimes I_m)\zeta + e) \\
&= (I \otimes A_r)(\beta^T \otimes I_m)\zeta \\
&= A_r a
\end{align*}
\]

for all \( t \in [b_k^i, b_{k+1}^i) \), with initial condition \( a(0) = (\beta^T \otimes I_m)\zeta(0) \). When \( t = b_k^i \), we have that \( a = a^- \).

Let \( \delta(t) = \zeta(t) - 1_N \otimes a(t) \). The norm of \( \delta(t) \) is a measure of the mismatch among the state variables \( \zeta_i \) of each agent,
at time $t$. From (14a) and (16), it follows that, for all $t \in [b_k, b_{k+1})$,
\[
\dot{\delta} = (I_N \otimes A_r)\zeta - (L \otimes I_m)(\zeta + e) - (\zeta + e) - N \otimes (A_r a) = (I_N \otimes A_r)\delta - (L \otimes I_m)(\delta + e).
\]  
(17)

When $t = b_k$, we have that $\dot{\delta} = \zeta - I_N \otimes a = \zeta - I_N \otimes a = \delta^\circ$. Note also that, using the properties of $\beta$, we obtain that, for all $t \geq 0$, $(\beta^T \otimes I_m)\delta = (\beta^T \otimes I_m)\zeta - (\beta^T \otimes I_m)\delta = - e - e = a - 1 \otimes a = 0$. In summary, $\delta$ satisfies
\[
\begin{cases}
    \dot{\delta} = Z\delta - (L \otimes I_m)e, & t \in [b_k, b_{k+1}), \\
    \delta = \delta^\circ, & t = b_k.
\end{cases}
\]  
(18a, 18b)

and $(\beta^T \otimes I_m)\delta(t) = 0$ for all $t \geq 0$. To derive a bound on the asymptotic behavior of $\delta$, we need the following lemma.

**Lemma 1.** Let $v \in \mathbb{R}^{N_m}$ be such that $(\beta^T \otimes I_m)v = 0$. If Assumption 1 holds, then there exist $\kappa \geq 1$ and $\lambda > 0$ such that, for all $t \geq 0$,
\[
\|e^{Z^T v}\| \leq \kappa e^{-\lambda t}\|v\|.
\]  
(19)

**Proof.** Let $\mathcal{L}$ be decomposed as in (1). Then, the matrix $Z$ defined in (15) may be written as
\[
Z = ([I_N \quad V] \otimes I_m) \text{diag}(A_r, Z)([\beta W^T] \otimes I_m)^T,
\]  
(20)

where $([\beta W^T] \otimes I_m)^T = ([I_N \quad V] \otimes I_m)^{-1}$ and $Z = I_{N-1} \otimes A_r - L \otimes I_m$.

(21)

It then follows that
\[
e^{Z^T v} = ([I_N \quad V] \otimes I_m)e^{\text{diag}(A_r, Z)t}([\beta W^T] \otimes I_m)^T v = (V \otimes I_m) e^{Z t} (W \otimes I_m)v,
\]  
(22)

where we used the fact that $(\beta^T \otimes I_m)v = 0$. Notice that $\sigma(Z) = \sigma(A_r) - \sigma(L)$ (see, e.g., [23, Theorem 4.4.5]), hence (4) implies that $Z$ is Hurwitz. Therefore, there exist $\kappa_1 \geq 1$ and $\lambda > 0$ such that, for all $t \geq 0$,
\[
\|e^{Z^T t}\| \leq \kappa_1 e^{-\lambda t}.
\]  
(23)

Using (22), (23), and the fact that $\|X \otimes I\| = \|X\|$ for any matrix $X$, we conclude that (19) is satisfied for $\kappa = \kappa_1 \|V\|/\|W\|$.

Lemma 1 is an extension of Lemma 2.1 in [18] that is recovered by considering only undirected connected graphs and taking $A_r = 0$ (in this case, we may set $\kappa = 1$ and $\lambda = \lambda_2(\mathcal{L})$). Using (18) and Lemma 1, we conclude the following.

**Theorem 2 (Theorem 2 for $A_r = 0$ and directed graphs).** If Assumption 1 holds, then, for all initial conditions $\zeta(0) \in \mathbb{R}^{N_m}$ and all $\alpha < \lambda$, the vector $\delta$ satisfies
\[
\|\delta(t)\| \leq \delta = \kappa \max \big\{\|\delta(0)\|, \bar{e}\big\},
\]  
(24)

for all $t \geq 0$, where $\bar{e} = \sqrt{\mathcal{N}}\|\mathcal{L}\|(c_0/\lambda + c_1/(\lambda - \alpha))$ and
\[
\lim_{t \to +\infty} \|\delta(t)\| \leq \delta_\infty = \frac{\kappa \sqrt{\mathcal{N}}\|\mathcal{L}\|}{\lambda} c_0.
\]  
(25)

Moreover, if $c_0 > 0$, then, for all $k \geq 0$ and all $i \in \{1, \ldots, N\}$,
\[
b_{k+1}^i - b_k^i \geq \theta_{\min} = \frac{1}{\omega} \log \left(1 + \frac{\omega c_0}{\theta}\right) > 0
\]  
(26)

where $\omega = \lambda_\max(A_r + \lambda R^2)/2$ and $\bar{v} = \|\mathcal{L}\|/(\delta + \sqrt{\mathcal{N}}(c_0 + c_1))$.

**Proof.** From (18), it follows that, for all $t \geq 0$,
\[
\delta(t) = e^{Z^T t} - \int_0^t e^{Z^T (t-s)} (L \otimes I_m) e(s) ds.
\]  
(27)

The triggering condition in (13) implies that, for all $t \geq 0$,
\[
\|e(t)\| = \sqrt{\sum_{i=1}^N \|e_i(t)\|^2} \leq \sqrt{\mathcal{N}} c(t).
\]  
(28)

Taking the norm in (27) and using Lemma 1, yields
\[
\|\delta(t)\| \leq \kappa e^{-\lambda t}\|\delta(0)\| + \int_0^t \kappa e^{-\lambda (t-s)} \|\mathcal{L} \otimes I_m\| e(s) ds \leq \kappa e^{-\lambda t}\|\delta(0)\| + \kappa \sqrt{\mathcal{N}}\|\mathcal{L}\| \left(\frac{c_0}{\lambda} + \frac{c_1}{\lambda - \alpha}(e^{-\lambda t} - e^{-\lambda T})\right).
\]  
(29)

If Zeno solutions are avoided, then the limit in (25) exists and $\delta_\infty$ is obtained from (29) by letting $t \to +\infty$. The bound in (24) is obtained by rewriting (29) as
\[
\|\delta(t)\| \leq \kappa \left\{e^{-\lambda t}\|\delta(0)\| - \bar{e} + \sqrt{\mathcal{N}}\|\mathcal{L}\| \left(\frac{c_0}{\lambda} + \frac{c_1}{\lambda - \alpha}(e^{-\lambda T} - e^{-\lambda t})\right)\right\}
\]  
(30)

and using the fact that $\max\{a - b, 0\} + b = \max\{a, b\}$.

To prove that the closed-loop system does not exhibit Zeno solutions, we show that the time interval between consecutive broadcasts of any agent is lower bounded by a positive number (this implies that the sequence $\{b_k\}_{k \geq 0}$ cannot have any accumulation points). Let $k \geq 0$ and $i \in \{1, \ldots, N\}$ be fixed. Using the fact that $e_i(b_k^i) = 0$, (12a) implies that
\[
e_i(t) = - \int_{b_k^i}^t e^{A_r(s)} v_i(s) ds,
\]  
(31)

for all $t \in [b_k^i, b_{k+1}^i)$. Applying norms on both sides, we obtain
\[
\|e_i(t)\| \leq \int_{b_k^i}^t \|e^{A_r(s)}\| \|v_i(s)\| ds \leq \int_{b_k^i}^t e^{\omega(s)} \|v_i(s)\| ds,
\]  
(32)

where we used the fact that $\omega$ is such that $\|e^{A_r(s)}\| \leq \omega t$ for all $t \geq 0$ (see, e.g., [24, Section 2]). Letting $v = (v_1, \ldots, v_N) = (\mathcal{L} \otimes I_m)(\zeta + e)$, we have that
\[
\|v_i\| \leq \|v\| = \|\mathcal{L} \otimes I_m\| \|\zeta + e\| = \|\mathcal{L} \otimes I_m\| (\delta + e) \leq \|\mathcal{L}\| (\|\delta\| + \|e\|) \leq \bar{v}.
\]  
(33)

Replacing (33) in (32) yields
\[
\|e_i(t)\| \leq \int_{b_k^i}^t e^{\omega(s)} \bar{v} ds = \frac{\bar{v}}{\omega} \left(e^{\omega(b_k^i)} - 1\right).
\]  
(34)
Hence, a lower bound on the minimum time interval between any two consecutive broadcast times of agent $i$ is given by the solution of
\[ \bar{v}(e^{\omega t - 1}) = \epsilon_0^\omega, \]
whose closed form is given in (26). Since $\theta_{\text{min}}$ is independent of both $k$ and $i$, the lower bound holds for all $k \geq 0$ and $i \in \{1, \ldots, N\}$. \hfill \Box

Notice that the asymptotic bound in (25) can be made arbitrarily small by decreasing $\epsilon_0$, albeit at the expense of making $\theta_{\text{min}}$ smaller as well. Also, both $\kappa$ and $\lambda$ depend on the weights assigned to each edge. Further study is required to analyze how to exploit this degree of freedom (weight assignment) to achieve some desired closed-loop properties.

C. Self-triggered communication protocol

To avoid spending computational resources by constantly testing if the broadcast condition has been violated, in this section we propose a self-triggered implementation of the event-triggered communication protocol defined in (13).

Suppose agent $\ell$ executes a broadcast at time $t = b_k$. Let $p_j = \max\{p \geq 0 : b_{pj}^t \leq b_k\}$ with $j \in \{1, \ldots, N\}$ denote the index of the last broadcast of agent $j$ (notice that $b_{pj}^t = b_k$). At this point, instead of continuously testing the event condition defined in (13) to determine the next broadcast time, agent $\ell$ computes $b_{p\ell+1}$ using the information available at the current time instant $b_k$. At the same time, all its out-neighbors have to recompute their next broadcast times as well to guarantee that their corresponding event conditions are satisfied. This is necessary because when $\zeta_\ell$ is updated, $v_j(b_k)$ changes for all $j \in N_\ell^+$ thereby altering the trajectory of $\zeta_\ell$ and $e_j$ for $t \geq b_k$.

In what follows, let $i \in \{\ell \cup N_\ell^+\}$. To derive an expression for the computation of $b_{p\ell+1}$ at time $t = b_k$, we start by solving (12a) in $t$, yielding
\[
e_i(t) = e^{A_i(t-b_k)}e_i(b_k) - \int_{b_k}^t e^{A_i(s-t)}v_i(s)ds,
\]
for all $t \in [b_k, b_{k+1})$. We have that $e_x(b_k) = 0$ but, in general, $e_j(b_k) \neq 0$ for $j \in N_\ell^+$. Given (35), finding a closed-form solution for the triggering condition $\|e_i(b_{p\ell+1}^t)\| = c(b_{p\ell+1}^t)$ is, in general, impossible. Instead of the exact solution, we will compute $b_{p\ell+1}^t$ such that $b_{p\ell+1}^t \leq b_{p\ell+1}$ thereby guaranteeing that $\|e_i(b_{p\ell+1}^t)\| \leq c(b_{p\ell+1})$ is satisfied. The goal is to keep the gap between $b_{p\ell+1}^t$ and $b_{p\ell+1}$ as small as possible. The self-triggered implementation is therefore expected to generate a sequence of broadcast times with a higher average broadcast rate than the one obtained in the event-triggered case.

To compute $b_{p\ell+1}^t$, we note that the dynamics of $\zeta_\ell$ given in (6) imply that, for all $t \in [b_k, b_{k+1})$, $\zeta_\ell(t) = e^{A_i(t-b_k)}\zeta_\ell(b_k)$. Thus, (9) may be written as $v_i(t) = e^{A_i(t-b_k)}\bar{v}_i(b_k)$ where
\[
\bar{v}_i(b_k) = \sum_{j=1}^N a_{ij}(\zeta_j(b_k) - \zeta_i(b_k)).
\]

Using this fact in (35) yields
\[
e_i(t) = e^{A_i(t-b_k)}e_i(b_k) - \int_{b_k}^t e^{A_i(s-t)}e^{A_i(s-b_k)}\bar{v}_i(b_k)ds
= e^{A_i(t-b_k)}e_i(b_k) - (t-b_k)e^{A_i(t-b_k)}\bar{v}_i(b_k),
\]
from which we obtain
\[
\|e_i(t)\| \leq e^{\omega (t-b_k)}(\|e_i(b_k)\| + \|\bar{v}_i(b_k)\|(t-b_k)).
\]
The next broadcast time is then defined as $b_{p\ell+1}^t = b_k + \theta_i^*$ where $\theta_i^*$ is the positive solution of
\[
e^{\omega t}(\|e_i(b_k)\| + \|\bar{v}_i(b_k)\|)/\theta = c(b_k + \theta).
\]

Note that $e_i(b_k)$ and $\bar{v}_i(b_k)$ are known to agent $i$ at time $t = b_k$, thus they may be used to compute the next broadcast time. Taking $c_1 = \alpha = \|e_i(b_k)\| = 0$ and using the fact that $\|\bar{v}_i(b_k)\| \leq \bar{v}$, a lower bound on the minimum broadcast interval of each agent is defined as the positive solution of
\[ \bar{v}\theta^e\theta = \epsilon_0 \] and denoted by $\theta_{\text{min}}^e$.

**Remark 1.** Solving (39) using a generic root finder may be time consuming. As an alternative, we propose a method that computes an approximation that is strictly smaller. Note that (39) may be written as
\[
\|\bar{v}_i(b_k)\|/\theta + \|e_i(b_k)\| = c_0 e^{-\omega \theta} + c_1 e^{-\alpha b_k} e^{-(\omega + \alpha) \theta},
\]
which is an equation of the form
\[
a x + b = c e^{-\alpha x} + d e^{-\beta x}
\]
where $a, b, c, d, \alpha, \beta \geq 0$. Let $x^*$ denote the unique positive solution of (41) (that exists if $b < c + d$). An approximation $x_1 < x^*$ is obtained by exploiting the convexity of the exponential terms in (41). For fixed $x_0 \geq 0$ and $\gamma \geq 0$, we have that $e^{-\gamma x} \geq e^{-\gamma x_0} (1 - \gamma (x - x_0))$ for all $x \geq 0$. Using this fact in (41), $x_1$ is defined as
\[
ax_1 + b = c(1 - \alpha (x_1 - x_0)) + d(1 - \beta(x_1 - x_0)),
\]
\[
\Leftrightarrow x_1 = \tilde{c}(1 + \alpha x_0) + \tilde{d}(1 + \beta x_0) - b
= \tilde{c}\alpha + \tilde{d}\beta + a,
\]
where $\bar{c} = c e^{-\alpha x_0}$ and $\bar{d} = d e^{-\beta x_0}$. A better approximation $x_1 < x^*$ is obtained by repeating this process, taking this time $x_0 = x_1$. Starting with $x_0 = 0$, this iterative process generates a strictly increasing sequence $\{x_k\}_{k \geq 0}$ that tends to $x^*$ from below, that is, for all $k \geq 0$, $x_k < x_{k+1} < x^*$ and $\lim_{k \to \infty} x_k = x^*$.

IV. EXAMPLE

In this section, we compare the proposed event-triggered and self-triggered communication protocols. We consider $N = 6$ agents with the dynamics of a fourth-order oscillator where
\[
A_r = \begin{bmatrix}
0 & -1 & 1 & 0 \\
1 & 0 & 0 & 1 \\
0 & 0 & 0 & -\frac{1}{3} \\
0 & 0 & \frac{1}{3} & 0
\end{bmatrix}.
\]

The agents exchange information according to the communication graph shown in Fig. 2a. Since $\sigma(A_r) \subset i\mathbb{R}$, Assumption 1 is satisfied. Selecting all edge weights equal to $\rho = 1$, we have that $\sigma(\mathcal{L}) = \{0, 0.5344, 1.5 \pm 0.8660, 2.2328 \pm i0.7926\}$, $\|\mathcal{L}\| = 2.9364$, and $\beta = (1/3, 1/3, 1/3, 0, 0, 0)$. The value of $\lambda$ is obtained by finding $P \succeq I_{m(N-1)}$ and $\lambda > 0$ such that $\mathcal{Z}^TP + P\mathcal{Z} + 2\lambda P \preceq 0$, where $\mathcal{Z}$ is defined in (21). Then, $\kappa$ is as defined in the proof of Lemma 1 with
The simulation results are presented in Fig. 2b-d. The trajectories of $\zeta_i$ when using event-triggered and self-triggered communication protocols are shown in Fig. 2b and Fig. 2c, respectively (in the latter case, we solved (39) using two iterations of the method presented in Remark 1). In both cases, the difference between the trajectories of any two agents is within a certain error tolerance, a fact that is corroborated by the trajectory of $||\delta||$ shown in Fig. 2d. The average sampling intervals observed were between 1.4416 s and 2.8512 s in the event-triggered case and between 1.2654 s and 1.6235 s in the self-triggered case, illustrating the conservativeness introduced in the derivation of the latter communication protocol.

V. CONCLUSIONS

In this paper, we proposed and analyzed a control architecture designed to achieve synchronization of a multi-agent system using event-triggered and self-triggered communication protocols. The proposed event-triggered communication protocol extends the work reported in [18] for event-triggered consensus, by allowing directed communication graphs and more general agent dynamics. We showed that the proposed control architecture achieves bounded synchronization errors and that the closed-system does not exhibit Zeno solutions.

REFERENCES