Synchronization of multi-agent systems using event-triggered and self-triggered broadcasts

João Almeida, Carlos Silvestre, and António M. Pascoal

Abstract—**This paper addresses the problem of synchronizing a group of identical linear time-invariant agents that exchange information through a communication network. The agents may only broadcast information at discrete time instants and the decision to execute a broadcast is based on an event-triggered communication protocol. We prove that with the proposed control architecture the state of each agent converges to and remains in a neighborhood of a desired reference signal and the closed-loop system does not exhibit Zeno solutions. A self-triggered implementation of the proposed event-triggered communication protocol is also derived.**

I. Introduction

In this paper, we define a multi-agent system as a dynamical system formed by a set of agents, each with dynamics modeled by a linear time-invariant (LTI) system, connected by a communication network that provides them with the means to exchange information. A survey of applications of multi-agent systems presented in [1] illustrates how local decentralized coordination strategies can be employed so that a desired global behavior is observed. A special class of applications requires the agents to align their states in a well-defined sense, with the most representative examples being the consensus and synchronization problems (see, e.g., [1]–[5]).

We address the synchronization problem for groups of identical agents. Although the authors of [5] solve this problem for groups of heterogeneous agents, our goal is to drop the assumption of continuous communication links present in [5] by employing sampled-data control techniques. The objective is to derive decentralized control laws and communication protocols capable of making the state of each agent converge to the same reference signal.

Due to the digital nature of the communication network, an additional constraint on the protocol design arises from the fact that communications can only occur at discrete time instants. The standard approach would be to broadcast information periodically. However, in recent years a different strategy has received attention due to a flurry of theoretical developments. Known as event-triggered control, in this new approach such tasks as sampling a signal or broadcasting information are only executed when deemed necessary according to some triggering conditions, often dependent on the state of each agent. For more details on this approach, the interested reader is referred to, e.g., [6]–[9] for the single plant case and to [10]–[12] for the case of multiple plants. It is important to point out that in event-triggered control, triggering conditions must be constantly monitored which may be infeasible for some applications. To circumvent this issue, self-triggered control strategies were developed where instead of continuously testing a triggering condition, an event scheduler computes when the next event should occur by using information available at the current time instant (see, e.g., [13]–[16]).

In a multi-agent scenario where agents have to communicate with each other, the event-triggered strategy is even more relevant since the communication medium is often shared by all agents, meaning that if each agent tried to transmit too often, successful communications would become impossible. Hence, by resorting to event-triggered control techniques, a communication protocol that avoids redundant broadcasts of information is sought. These techniques have been applied to the consensus problem in [17]–[20]. We note that the consensus problem is a particular type of synchronization problem where the reference signal is constant.

The contribution of this paper is twofold: i) we extend the event-triggered consensus results reported in [18] for 1st and 2nd order integrators with an undirected communication network; this is done by deriving an event-triggered communication protocol capable of achieving synchronization for a class of agents with LTI dynamics that are connected by a directed communication network and ii) we offer a self-triggered implementation of the proposed event-triggered communication protocol.

Notation: If $\{a_k\}_{k\geq 0}$ and $\{b_k\}_{k\geq 0}$ are two strictly increasing sequences with elements in $\mathbb{R}$, then their union is a sequence $\{c_k\}_{k\geq 0}$ defined as the set of unique elements in $\{a_k\}_{k\geq 0}$ and $\{b_k\}_{k\geq 0}$ reordered to satisfy $c_k < c_{k+1}$ for all $k \geq 0$. We denote this by writing $\{c_k\}_{k\geq 0} = \{a_k\}_{k\geq 0} \cup \{b_k\}_{k\geq 0}$. For a complex number z, $\Re\{z\}$ denotes its real part. For a signal $x : [0, +\infty) \to \mathbb{R}^n$, if the limit from below at time $t \in [0, +\infty)$ exists, then it is defined as $x^-(t) = \lim_{s\uparrow t} x(s)$. If t is understood from context, we simply write x and x^- to stand for $x(t)$ and $x^-(t)$, respectively. A vector of dimension n whose entries are all equal to one is denoted by $\mathbf{1}_n$. Given a collection of vectors $\{x_1, \ldots, x_N\}$ where $x_i \in \mathbb{R}^{n_i}$, the vector obtained by stacking all x_i column-wise is represented by $z = (x_1, \ldots, x_N) = \begin{bmatrix} x_1^\top & \ldots & x_N^\top \end{bmatrix}^\top$. The symbol I_n

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denotes the identity matrix of dimension n. For a square matrix X, e^X, $\|X\|$, and $\sigma(X)$ denote its matrix exponential, its spectral norm (defined as its largest singular value), and its spectrum (the set of eigenvalues of X), respectively. The symbol $\otimes$ denotes the Kronecker product.

II. Graph theory review

In this section we introduce some necessary concepts and results from graph theory (adapted from [1], [21]) required for the presentation and analysis of our proposed solution for the problem of event-triggered synchronization.

A (directed) graph $\mathcal{G} = \mathcal{G}(\mathcal{V}, \mathcal{E})$ consists of a finite set $\mathcal{V} = \{1, 2, \ldots, N\}$ of N vertices and a finite set $\mathcal{E} \subseteq \mathcal{V} \times \mathcal{V}$ of m ordered pairs of vertices (i, j) named edges (in this paper, self-edges (i, i) are not allowed). An undirected graph is defined as a graph where $(i, j) \in \mathcal{E}$ if and only if $(j, i) \in \mathcal{E}$. If $(i, j) \in \mathcal{E}$, then we say that vertex i is an in-neighbor of vertex j and that j is an out-neighbor of vertex i. The set of in-neighbors and the set of out-neighbors of vertex i are defined as $\mathcal{N}_i^- = \{j \in \mathcal{V} : (j, i) \in \mathcal{E}\}$ and $\mathcal{N}_i^+ = \{j \in \mathcal{V} : (i, j) \in \mathcal{E}\}$, respectively. A path in $\mathcal{G}$ from vertex i to vertex j is a sequence of distinct edges of the form $\{(i, i_1), (i_1, i_2), \ldots, (i_k, j)\}$. A vertex i is a root of a graph $\mathcal{G}$ if there exists a path in $\mathcal{G}$ from vertex i to every other vertex in $\mathcal{G}$. If $\mathcal{G}$ has at least one root, we say that it is a rooted graph. If a graph $\mathcal{G}$ is undirected and rooted, then it is said to be connected (in this case, all vertices are roots). A weighted graph is a graph where a real number (weight) is associated with every edge in the graph (in this paper, all graphs are weighted). The adjacency matrix of a graph, denoted by $\mathcal{A} = [a_{ij}] \in \mathbb{R}^{N\times N}$, is a square matrix with rows and columns indexed by the vertices and whose entries satisfy $a_{ij} > 0$ if $(j, i) \in \mathcal{E}$ and are zero otherwise (a_{ij} denotes the weight for edge $(j, i) \in \mathcal{E}$). The in-degree matrix $\mathcal{D}$ of a graph is a diagonal matrix where the (i, i)-entry is equal to the in-degree of vertex i defined as $\sum_{j=1}^N a_{ij}$. The Laplacian matrix of a graph $\mathcal{G}$ is defined as $\mathcal{L} = \mathcal{D} - \mathcal{A}$ and has the following properties: i) $\mathcal{L}\mathbf{1}_N = 0$ and there exists $\beta \in \mathbb{R}^N, \beta^\top \mathbf{1}_N = 1$ such that $\beta^\top\mathcal{L} = 0$; ii) $\sigma(\mathcal{L}) = \{0, \lambda_2, \ldots, \lambda_N\}$ with $\Re\{\lambda_i\} > 0$ for all non-zero eigenvalues; iii) $\mathcal{G}$ is a rooted graph if and only if 0 is a simple eigenvalue of $\mathcal{L}$; iv) if $\mathcal{G}$ is a rooted graph, then there exist matrices $\hat{\mathcal{L}} \in \mathbb{R}^{(N-1)\times(N-1)}$, $V \in \mathbb{R}^{N\times(N-1)}$, and $W \in \mathbb{R}^{(N-1)\times N}$ such that $\sigma(\hat{\mathcal{L}}) = \sigma(\mathcal{L})\backslash\{0\}$, $\begin{bmatrix}\mathbf{1}_N & V\end{bmatrix}$ is nonsingular, $\begin{bmatrix}\beta & W^\top\end{bmatrix}^\top = \begin{bmatrix}\mathbf{1}_N & V\end{bmatrix}^{-1}$, and

$$\mathcal{L} = \begin{bmatrix}\mathbf{1}_N & V\end{bmatrix} \operatorname{diag}(0, \hat{\mathcal{L}}) \begin{bmatrix}\beta & W^\top\end{bmatrix}^\top. \tag{1}$$

III. Synchronization of multi-agent systems

The multi-agent system that we consider consists of N agents with identical LTI dynamics. Each agent has a state denoted by $\zeta_i \in \mathbb{R}^m$ such that $\zeta_i(0) \in \mathbb{R}^m$ and, for all $t \geq 0$,

$$\dot{\zeta}_i = A_r\zeta_i + v_i \tag{2}$$

where $v_i \in \mathbb{R}^m$ is the control input and $A_r \in \mathbb{R}^{m\times m}$ (A_r may have unstable eigenvalues). To achieve synchronization, the state ζ_i must evolve in such a way that its trajectory is eventually the same across all agents. Note that due to different initial conditions, the agents are not guaranteed to converge to the same trajectory. In order to correct this misalignment, the agents must exchange information among them by using a communication network. This network is modeled by a communication graph $\mathcal{G}$ that is assumed to be fixed over time and where an edge $(j, i) \in \mathcal{E}$ means that agent i receives information from agent j. Convergence to the same trajectory must be distributed in the sense that v_i may only depend on information obtained from the in-neighbors of agent i. When continuous communication links among agents are allowed, one possible solution is to let v_i be given by

$$v_i = \sum_{j=1}^N a_{ij}(\zeta_j - \zeta_i), \tag{3}$$

where a_{ij} denotes the entries of the adjacency matrix associated with $\mathcal{G}$. In [3] it is shown that if the following assumption is satisfied, then all agents synchronize asymptotically, that is, for all initial conditions $\zeta_i(0) \in \mathbb{R}^m$, $\lim_{t\to+\infty} \|\zeta_i(t) - \zeta_j(t)\| = 0$, for all $i, j \in \{1, \ldots, N\}$.

Assumption 1. The communication graph $\mathcal{G}$ is a rooted graph and, for all $i \in \{1, \ldots, m\}$ and $j \in \{2, \ldots, N\}$,

$$\Re\{\lambda_i(A_r) - \lambda_j(\mathcal{L})\} < 0. \tag{4}$$

Assumption 1 requires the connectivity of the graph to be strong enough to dominate the unstable dynamics in A_r so that (4) holds. One way to accomplish this is to design the graph topology or edge weights such that the non-zero eigenvalues of $\mathcal{L}$ meet condition (4). This is always possible, as shown next. Suppose $\mathcal{G}_\rho$ is a rooted graph with all edge weights equal to $\rho > 0$ and let $\mathcal{L}_\rho$ denote its corresponding Laplacian matrix. Notice that $\mathcal{L}_\rho = \rho\mathcal{L}_1$. In this case, the stability condition (4) becomes

$$\rho > \max_{\substack{i,j\\ j\neq 1}} \frac{\Re\{\lambda_i(A_r)\}}{\Re\{\lambda_j(\mathcal{L}_1)\}} = \frac{\max_i \Re\{\lambda_i(A_r)\}}{\min_{j\neq 1} \Re\{\lambda_j(\mathcal{L}_1)\}}. \tag{5}$$

Thus, by selecting ρ sufficiently large, the unstable dynamics of A_r (eigenvalues of A_r with positive real part) can be dominated. Note that if all eigenvalues of A_r are imaginary, that is, $\sigma(A_r) \subset \mathrm{i}\mathbb{R}$, then (4) is satisfied for all rooted graphs.

A. Event-triggered synchronization

To avoid the need for continuous communication links in (3) and inspired by the work reported in [18] for event-triggered consensus, in this section we propose an event-triggered solution for the multi-agent synchronization problem.

The proposed control architecture is represented in Fig. 1, from the point of view of agent i. The agent has been augmented with additional state variables and is responsible for deciding when its current state should be broadcast to the network, as represented by the broadcast event detector. This event detector triggers the broadcast of the current value of ζ_i to the out-neighbors of agent i whenever a given state dependent condition is violated. The sequence of time instants where these violations occur is referred to as the sequence of broadcast times of agent i and is denoted by $\{b_k^i\}_{k\geq 0}$.

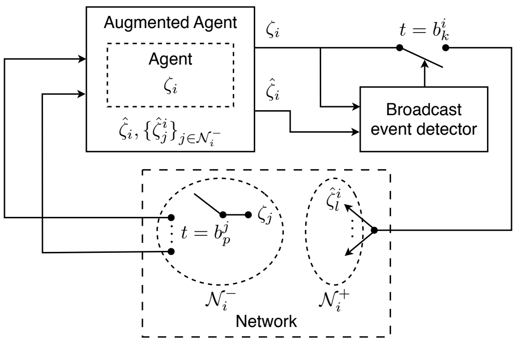

Fig. 1. Proposed control architecture from the point of view of agent i.

The state of the augmented agent is described by three variables: ζ_i, $\hat{\zeta}_i$, and $\hat{\zeta}_j^i$ with $j \in \mathcal{N}_i^-$. The dynamics of ζ_i are as in (2). The state variable $\hat{\zeta}_i$ evolves according to a unperturbed model between broadcast times of agent i and is reset to the current value of ζ_i when a broadcast occurs. The dynamics of $\hat{\zeta}_i$ may be written in the form of an impulsive system as

$$\begin{cases} \dot{\hat{\zeta}}_i = A_r \hat{\zeta}_i, & t \in [b_k^i, b_{k+1}^i), \quad (6a) \\ \hat{\zeta}_i = \zeta_i^-, & t = b_k^i \quad (6b) \end{cases}$$

(for an introduction to impulsive systems see, e.g., [22]). The additional states $\hat{\zeta}_j^i$ represent local replicas of the state ζ_j of agent i's in-neighbors and are used to store information received from them. When an in-neighbor of agent i, say $j \in \mathcal{N}_i^-$, broadcasts the current value of ζ_j, this value is used to reset the value of $\hat{\zeta}_j^i$, as modeled by the impulsive system

$$\begin{cases} \dot{\hat{\zeta}}_j^i = A_r \hat{\zeta}_j^i, & t \in [b_k^j, b_{k+1}^j), \quad (7a) \\ \hat{\zeta}_j^i = \zeta_j^-, & t = b_k^j\ . \quad (7b) \end{cases}$$

To remove the need for continuous communication links among agents, (3) is replaced by

$$v_i = \sum_{j=1}^{N} a_{ij}(\hat{\zeta}_j^i - \hat{\zeta}_i). \tag{8}$$

Without loss of generality, suppose that, for all $i, j \in \{1, \ldots, N\}$, $\hat{\zeta}_j^i$ is initialized with the value $\zeta_j(0)$. Since $\hat{\zeta}_j$ and $\hat{\zeta}_j^i$ have the same dynamics (compare (6) with $i = j$ and (7)), we have that $\hat{\zeta}_j^i(t) = \hat{\zeta}_j(t)$ for all $t \geq 0$. Therefore, for analysis purposes, only the state variables $\hat{\zeta}_j$ are required and we may write (8) as

$$v_i = \sum_{j=1}^{N} a_{ij}(\hat{\zeta}_j - \hat{\zeta}_i). \tag{9}$$

If the broadcasted information were to arrive at each out-neighbor of agent i at different times due to, e.g., transmission delays, then the previous simplification would not be possible.

Finally, the sequence of broadcast times satisfies

$$b_{k+1}^i = \inf\{t > b_k^i : \|\hat{\zeta}_i(t) - \zeta_i(t)\| = c(t)\}, \tag{10}$$

for all $k \geq 0$, where $b_0^i = 0$ and $c(t)$ represents a time-varying threshold defined as $c(t) = c_0 + c_1 \mathrm{e}^{-\alpha t}$ with $c_0, c_1, \alpha \geq 0$.

The setup described in [18] for event-triggered consensus is recovered by taking $m = 1$ and $A_r = 0$. In this case, the authors proved the following result.

Theorem 1 ([18, Theorem 3.2]). *If $\mathcal{G}$ is an undirected connected graph and $c_0 > 0$, then the closed-loop system does not exhibit Zeno solutions and each agent's trajectory satisfies*

$$\lim_{t\to+\infty} \|\zeta_i(t) - a\| \leq \frac{\sqrt{N}\|\mathcal{L}\|}{\lambda_2(\mathcal{L})} c_0, \tag{11}$$

for all $\zeta_i(0) \in \mathbb{R}$, where $a = \frac{1}{N}\sum_{i=1}^{N} \zeta_i(0)$ and $\lambda_2(\mathcal{L})$ is the smallest nonzero eigenvalue of $\mathcal{L}$.

In the next section, we extend Theorem 1 by allowing directed graphs and an arbitrary A_r matrix as long as Assumption 1 is satisfied.

B. Stability analysis

For analysis purposes, it is more convenient to work with the errors $e_i = \hat{\zeta}_i - \zeta_i$ that originate from the fact that $\hat{\zeta}_i$ is used for feedback rather than ζ_i. The dynamics of e_i are given by

$$\begin{cases} \dot{e}_i = A_r e_i - v_i, & t \in [b_k^i, b_{k+1}^i), \quad (12a) \\ e_i = 0, & t = b_k^i. \quad (12b) \end{cases}$$

Using the error e_i, (10) is equivalent to

$$b_{k+1}^i = \inf\{t > b_k^i : \|e_i(t)\| = c(t)\}. \tag{13}$$

Let $\zeta = (\zeta_1, \ldots, \zeta_N)$ and $e = (e_1, \ldots, e_N)$ denote new state vectors. Their dynamics are derived from (2), (9), and (12), and may be written as

$$\begin{cases} \begin{bmatrix} \dot{\zeta} \\ \dot{e} \end{bmatrix} = \begin{bmatrix} Z & -\mathcal{L} \otimes I_m \\ \mathcal{L} \otimes I_m & I_N \otimes A_r + \mathcal{L} \otimes I_m \end{bmatrix} \begin{bmatrix} \zeta \\ e \end{bmatrix}, \ t \in [b_k, b_{k+1}), & (14a) \\ \begin{bmatrix} \zeta \\ e \end{bmatrix} = \begin{bmatrix} I_N \otimes I_m & 0 \\ 0 & (I_N - R_k) \otimes I_m \end{bmatrix} \begin{bmatrix} \zeta^- \\ e^- \end{bmatrix}, \quad t = b_k, & (14b) \end{cases}$$

where $\{b_k\}_{k\geq 0} = \bigcup_{i=1}^{N} \{b_k^i\}_{k\geq 0}$,

$$Z = I_N \otimes A_r - \mathcal{L} \otimes I_m, \tag{15}$$

and $R_k = \mathrm{diag}(r_{1,k}, r_{2,k}, \ldots, r_{N,k})$ is a diagonal matrix whose entries satisfy $r_{i,k} = 1$ if $b_p^i = b_k$ for some $p \geq 0$ and are zero otherwise.

We will show that each ζ_i converges to a neighborhood of the reference signal $a(t) = (\beta^\top \otimes I_m)\zeta(t)$. Note that if the graph is undirected, then $\mathcal{L}$ is symmetric, $\beta = \mathbf{1}_N/N$, and $a(t)$ becomes the average of all $\zeta_i(t)$. The signal a satisfies

$$\begin{aligned} \dot{a} &= (\beta^\top \otimes I_m)\left(Z\zeta - (\mathcal{L} \otimes I_m)e\right) \\ &= (\beta^\top \otimes A_r)\zeta - ((\beta^\top \mathcal{L}) \otimes I_m)(\zeta + e) \\ &= (1 \otimes A_r)(\beta^\top \otimes I_m)\zeta \\ &= A_r a, \end{aligned} \tag{16}$$

for all $t \in [b_k, b_{k+1})$, with initial condition $a(0) = (\beta^\top \otimes I_m)\zeta(0)$. When $t = b_k$, we have that $a = a^-$.

Let $\delta(t) = \zeta(t) - \mathbf{1}_N \otimes a(t)$. The norm of $\delta(t)$ is a measure of the mismatch among the state variables ζ_i of each agent, at time t. From (14a) and (16), it follows that, for all $t \in [b_k, b_{k+1})$,

$$\begin{aligned}\dot{\delta} &= (I_N \otimes A_r)\zeta - (\mathcal{L} \otimes I_m)(\zeta + e) - \mathbf{1}_N \otimes (A_r a) \\ &= (I_N \otimes A_r)(\zeta - \mathbf{1}_N \otimes a) - (\mathcal{L} \otimes I_m)(\delta + \mathbf{1}_N \otimes a + e) \\ &= (I_N \otimes A_r)\delta - (\mathcal{L} \otimes I_m)(\delta + e). \end{aligned} \tag{17}$$

When $t = b_k$, we have that $\delta = \zeta - \mathbf{1}_N \otimes a = \zeta^- - \mathbf{1}_N \otimes a^- = \delta^-$. Note also that, using the properties of β, we obtain that, for all $t \geq 0$, $(\beta^\top \otimes I_m)\delta = (\beta^\top \otimes I_m)\zeta - (\beta^\top \mathbf{1}_N \otimes a) = a - 1 \otimes a = 0$. In summary, δ satisfies

$$\begin{cases} \dot{\delta} = Z\delta - (\mathcal{L} \otimes I_m)e, & t \in [b_k, b_{k+1}), \quad \text{(18a)} \\ \delta = \delta^-, & t = b_k, \quad \text{(18b)} \end{cases}$$

and $(\beta^\top \otimes I_m)\delta(t) = 0$ for all $t \geq 0$. To derive a bound on the asymptotic behavior of δ, we need the following lemma.

Lemma 1. *Let $v \in \mathbb{R}^{Nm}$ be such that $(\beta^\top \otimes I_m)v = 0$. If Assumption 1 holds, then there exist $\kappa \geq 1$ and $\lambda > 0$ such that, for all $t \geq 0$,*

$$\|\mathrm{e}^{Zt} v\| \leq \kappa \mathrm{e}^{-\lambda t} \|v\|. \tag{19}$$

Proof. Let $\mathcal{L}$ be decomposed as in (1). Then, the matrix Z defined in (15) may be written as

$$Z = \left(\begin{bmatrix} \mathbf{1}_N & V \end{bmatrix} \otimes I_m \right) \operatorname{diag}(A_r, \hat{Z}) \left(\begin{bmatrix} \beta & W^\top \end{bmatrix} \otimes I_m \right)^\top, \tag{20}$$

where $\left(\begin{bmatrix} \beta & W^\top \end{bmatrix} \otimes I_m \right)^\top = \left(\begin{bmatrix} \mathbf{1}_N & V \end{bmatrix} \otimes I_m \right)^{-1}$ and

$$\hat{Z} = I_{N-1} \otimes A_r - \hat{\mathcal{L}} \otimes I_m. \tag{21}$$

It then follows that

$$\begin{aligned}\mathrm{e}^{Zt} v &= \left(\begin{bmatrix} \mathbf{1}_N & V \end{bmatrix} \otimes I_m \right) \mathrm{e}^{\operatorname{diag}(A_r, \hat{Z})t} \left(\begin{bmatrix} \beta & W^\top \end{bmatrix} \otimes I_m \right)^\top v \\ &= (V \otimes I_m)\, \mathrm{e}^{\hat{Z}t} (W \otimes I_m)\, v, \end{aligned} \tag{22}$$

where we used the fact that $(\beta^\top \otimes I_m)v = 0$. Notice that $\sigma(\hat{Z}) = \sigma(A_r) - \sigma(\hat{\mathcal{L}})$ (see, e.g., [23, Theorem 4.4.5]), hence (4) implies that $\hat{Z}$ is Hurwitz. Therefore, there exist $\kappa_1 \geq 1$ and $\lambda > 0$ such that, for all $t \geq 0$,

$$\|\mathrm{e}^{\hat{Z}t}\| \leq \kappa_1 \mathrm{e}^{-\lambda t}. \tag{23}$$

Using (22), (23), and the fact that $\|X \otimes I\| = \|X\|$ for any matrix X, we conclude that (19) is satisfied for $\kappa = \kappa_1 \|V\| \|W\|$. □

Lemma 1 is an extension of Lemma 2.1 in [18] that is recovered by considering only undirected connected graphs and taking $A_r = 0$ (in this case, we may set $\kappa = 1$ and $\lambda = \lambda_2(\mathcal{L})$). Using (18) and Lemma 1, we conclude the following.

Theorem 2 (Theorem 1 for $A_r \neq 0$ and directed graphs)**.** *If Assumption 1 holds, then, for all initial conditions $\zeta(0) \in \mathbb{R}^{Nm}$ and all $\alpha < \lambda$, the vector δ satisfies*

$$\|\delta(t)\| \leq \bar{\delta} = \kappa \max\left\{ \|\delta(0)\|, \bar{c} \right\}, \tag{24}$$

for all $t \geq 0$, where $\bar{c} = \sqrt{N}\|\mathcal{L}\|(c_0/\lambda + c_1/(\lambda - \alpha))$ and

$$\lim_{t \to +\infty} \|\delta(t)\| \leq \bar{\delta}_\infty = \frac{\kappa \sqrt{N} \|\mathcal{L}\|}{\lambda} c_0. \tag{25}$$

Moreover, if $c_0 > 0$, then, for all $k \geq 0$ and all $i \in \{1, \ldots, N\}$,

$$b^i_{k+1} - b^i_k \geq \theta_{\min} = \frac{1}{\omega} \log\left(1 + \frac{\omega c_0}{\bar{v}}\right) > 0 \tag{26}$$

where $\omega = \lambda_{\max}(A_r + A_r^\top)/2$ and $\bar{v} = \|\mathcal{L}\|(\bar{\delta} + \sqrt{N}(c_0 + c_1))$.

Proof. From (18), it follows that, for all $t \geq 0$,

$$\delta(t) = \mathrm{e}^{Zt}\delta(0) - \int_0^t \mathrm{e}^{Z(t-s)} (\mathcal{L} \otimes I_m) e(s) \mathrm{d}s. \tag{27}$$

The triggering condition in (13) implies that, for all $t \geq 0$,

$$\|e(t)\| = \sqrt{\sum_{i=1}^N \|e_i(t)\|^2} \leq \sqrt{N} c(t). \tag{28}$$

Taking the norm in (27) and using Lemma 1, yields

$$\begin{aligned}\|\delta(t)\| &\leq \kappa \mathrm{e}^{-\lambda t} \|\delta(0)\| + \int_0^t \kappa \mathrm{e}^{-\lambda(t-s)} \|(\mathcal{L} \otimes I_m) e(s)\| \mathrm{d}s \\ &\overset{(28)}{\leq} \kappa \mathrm{e}^{-\lambda t} \|\delta(0)\| + \int_0^t \kappa \mathrm{e}^{-\lambda(t-s)} \|\mathcal{L}\| \sqrt{N} c(s) \mathrm{d}s \\ &\leq \kappa \mathrm{e}^{-\lambda t} \|\delta(0)\| + \kappa \sqrt{N} \|\mathcal{L}\| \Big(\frac{c_0}{\lambda} \left(1 - \mathrm{e}^{-\lambda t}\right) \\ &\quad + \frac{c_1}{\lambda - \alpha} \left(\mathrm{e}^{-\alpha t} - \mathrm{e}^{-\lambda t} \right) \Big). \end{aligned} \tag{29}$$

If Zeno solutions are avoided, then the limit in (25) exists and $\bar{\delta}_\infty$ is obtained from (29) by letting $t \to +\infty$. The bound in (24) is obtained by rewriting (29) as

$$\begin{aligned}\|\delta(t)\| &\leq \kappa \Big\{ \mathrm{e}^{-\lambda t} (\|\delta(0)\| - \bar{c}) + \sqrt{N} \|\mathcal{L}\| \Big(\frac{c_0}{\lambda} + \frac{c_1}{\lambda - \alpha} \mathrm{e}^{-\alpha t} \Big) \Big\} \\ &\leq \kappa \{ \max\{ \|\delta(0)\| - \bar{c}, 0 \} + \bar{c} \}, \end{aligned} \tag{30}$$

and using the fact that $\max\{a - b, 0\} + b = \max\{a, b\}$.

To prove that the closed-loop system does not exhibit Zeno solutions, we show that the time interval between consecutive broadcasts of any agent is lower bounded by a positive number (this implies that the sequence $\{b_k\}_{k \geq 0}$ cannot have any accumulation points). Let $k \geq 0$ and $i \in \{1, \ldots, N\}$ be fixed. Using the fact that $e_i(b^i_k) = 0$, (12a) implies that

$$e_i(t) = -\int_{b^i_k}^t \mathrm{e}^{A_r(t-s)} v_i(s) \mathrm{d}s, \tag{31}$$

for all $t \in [b^i_k, b^i_{k+1})$. Applying norms on both sides, we obtain

$$\|e_i(t)\| \leq \int_{b^i_k}^t \|\mathrm{e}^{A_r(t-s)}\| \|v_i(s)\| \mathrm{d}s \leq \int_{b^i_k}^t \mathrm{e}^{\omega(t-s)} \|v_i(s)\| \mathrm{d}s, \tag{32}$$

where we used the fact that ω is such that $\|\mathrm{e}^{A_r t}\| \leq \mathrm{e}^{\omega t}$ for all $t \geq 0$ (see, e.g., [24, Section 2]). Letting $v = (v_1, \ldots, v_N) = (\mathcal{L} \otimes I_m)(\zeta + e)$, we have that

$$\begin{aligned}\|v_i\| \leq \|v\| &= \|(\mathcal{L} \otimes I_m)(\zeta + e)\| = \|(\mathcal{L} \otimes I_m)(\delta + e)\| \\ &\leq \|\mathcal{L}\|(\|\delta\| + \|e\|) \leq \bar{v}. \end{aligned} \tag{33}$$

Replacing (33) in (32) yields

$$\|e_i(t)\| \leq \int_{b^i_k}^t \mathrm{e}^{\omega(t-s)} \bar{v} \mathrm{d}s = \frac{\bar{v}}{\omega} \left(\mathrm{e}^{\omega(t - b^i_k)} - 1 \right). \tag{34}$$

Hence, a lower bound on the minimum time interval between any two consecutive broadcast times of agent i is given by the solution of $\bar{v}\left(\mathrm{e}^{\omega\theta}-1\right)=c_0\omega$, whose closed form is given in (26). Since $\theta_{\min}$ is independent of both k and i, the lower bound holds for all $k\geq 0$ and $i\in\{1,\ldots,N\}$. □

Notice that the asymptotic bound in (25) can be made arbitrarily small by decreasing c_0, albeit at the expense of making $\theta_{\min}$ smaller as well. Also, both κ and λ depend on the weights assigned to each edge. Further study is required to analyze how to exploit this degree of freedom (weight assignment) to achieve some desired closed-loop properties.

C. Self-triggered communication protocol

To avoid spending computational resources by constantly testing if the broadcast condition has been violated, in this section we propose a self-triggered implementation of the event-triggered communication protocol defined in (13).

Suppose agent ℓ executes a broadcast at time $t=b_k$. Let $p_j=\max\{p\geq 0: b_p^j\leq b_k\}$ with $j\in\{1,\ldots,N\}$ denote the index of the last broadcast of agent j (notice that $b_{p_\ell}^\ell=b_k$). At this point, instead of continuously testing the event condition defined in (13) to determine the next broadcast time, agent ℓ computes $b_{p_\ell+1}^\ell$ using the information available at the current time instant b_k. At the same time, all its out-neighbors have to recompute their next broadcast times as well to guarantee that their corresponding event conditions are satisfied. This is necessary because when $\hat{\zeta}_\ell$ is updated, $v_j(b_k)$ changes for all $j\in\mathcal{N}_\ell^+$ thereby altering the trajectory of ζ_j and e_j for $t\geq b_k$.

In what follows, let $i\in\{\ell\}\cup\mathcal{N}_\ell^+$. To derive an expression for the computation of $b_{p_i+1}^i$ at time $t=b_k$, we start by solving (12a) in t, yielding

$$e_i(t)=\mathrm{e}^{A_r(t-b_k)}e_i(b_k)-\int_{b_k}^{t}\mathrm{e}^{A_r(t-s)}v_i(s)\mathrm{d}s, \tag{35}$$

for all $t\in[b_k,b_{k+1})$. We have that $e_\ell(b_k)=0$ but, in general, $e_j(b_k)\neq 0$ for $j\in\mathcal{N}_\ell^+$. Given (35), finding a closed-form solution for the triggering condition $\|e_i(b_{p_i+1}^{i,*})\|=c(b_{p_i+1}^{i,*})$ is, in general, impossible. Instead of the exact solution, we will compute $b_{p_i+1}^i$ such that $b_{p_i+1}^i\leq b_{p_i+1}^{i,*}$ thereby guaranteeing that $\|e_i(b_{p_i+1}^i)\|\leq c(b_{p_i+1}^i)$ is satisfied. The goal is to keep the gap between $b_{p_i+1}^i$ and $b_{p_i+1}^{i,*}$ as small as possible. The self-triggered implementation is therefore expected to generate a sequence of broadcast times with an higher average broadcast rate than the one obtained in the event-triggered case.

To compute $b_{p_i+1}^i$, we note that the dynamics of $\hat{\zeta}_i$ given in (6) imply that, for all $t\in[b_k,b_{k+1})$, $\hat{\zeta}_i(t)=\mathrm{e}^{A_r(t-b_k)}\zeta_i(b_k)$. Thus, (9) may be written as $v_i(t)=\mathrm{e}^{A_r(t-b_k)}\bar{v}_i(b_k)$ where

$$\bar{v}_i(b_k)=\sum_{j=1}^{N}a_{ij}(\hat{\zeta}_j(b_k)-\hat{\zeta}_i(b_k)). \tag{36}$$

Using this fact in (35) yields

$$\begin{aligned} e_i(t)&=\mathrm{e}^{A_r(t-b_k)}e_i(b_k)-\int_{b_k}^{t}\mathrm{e}^{A_r(t-s)}\mathrm{e}^{A_r(s-b_k)}\bar{v}_i(b_k)\mathrm{d}s\\ &=\mathrm{e}^{A_r(t-b_k)}e_i(b_k)-(t-b_k)\mathrm{e}^{A_r(t-b_k)}\bar{v}_i(b_k), \end{aligned} \tag{37}$$

from which we obtain

$$\|e_i(t)\|\leq\mathrm{e}^{\omega(t-b_k)}\left(\|e_i(b_k)\|+\|\bar{v}_i(b_k)\|(t-b_k)\right). \tag{38}$$

The next broadcast time is then defined as $b_{p_i+1}^i=b_k+\theta_i^*$ where θ_i^* is the positive solution of

$$\mathrm{e}^{\omega\theta}(\|e_i(b_k)\|+\|\bar{v}_i(b_k)\|\theta)=c(b_k+\theta). \tag{39}$$

Note that $e_i(b_k)$ and $\bar{v}_i(b_k)$ are known to agent i at time $t=b_k$, thus they may be used to compute the next broadcast time. Taking $c_1=\alpha=\|e_i(b_k)\|=0$ and using the fact that $\|\bar{v}_i(b_k)\|\leq\bar{v}$, a lower bound on the minimum broadcast interval of each agent is defined as the positive solution of $\bar{v}\theta\mathrm{e}^{\omega\theta}=c_0$ and denoted by $\theta_{\min}^{\text{self}}$.

Remark 1. Solving (39) using a generic root finder may be time consuming. As an alternative, we propose a method that computes an approximation that is strictly smaller. Note that (39) may be written as

$$\|\bar{v}_i(b_k)\|\theta+\|e_i(b_k)\|=c_0\mathrm{e}^{-\omega\theta}+c_1\mathrm{e}^{-\alpha b_k}\mathrm{e}^{-(\omega+\alpha)\theta}, \tag{40}$$

which is an equation of the form

$$ax+b=c\mathrm{e}^{-\alpha x}+d\mathrm{e}^{-\beta x} \tag{41}$$

where $a,b,c,d,\alpha,\beta\geq 0$. Let x^* denote the unique positive solution of (41) (that exists if $b<c+d$). An approximation $x_1<x^*$ is obtained by exploiting the convexity of the exponential terms in (41). For fixed $x_0\geq 0$ and $\gamma\geq 0$, we have that $\mathrm{e}^{-\gamma x}\geq\mathrm{e}^{-\gamma x_0}(1-\gamma(x-x_0))$ for all $x\geq 0$. Using this fact in (41), x_1 is defined as

$$ax_1+b=\tilde{c}(1-\alpha(x_1-x_0))+\tilde{d}(1-\beta(x_1-x_0)) \tag{42}$$

$$\Leftrightarrow x_1=\frac{\tilde{c}(1+\alpha x_0)+\tilde{d}(1+\beta x_0)-b}{\tilde{c}\alpha+\tilde{d}\beta+a}, \tag{43}$$

where $\tilde{c}=c\mathrm{e}^{-\alpha x_0}$ and $\tilde{d}=d\mathrm{e}^{-\beta x_0}$. A better approximation $x_1<x_2<x^*$ is obtained by repeating this process, taking this time $x_0=x_1$. Starting with $x_0=0$, this iterative process generates a strictly increasing sequence $\{x_k\}_{k\geq 0}$ that tends to x^* from below, that is, for all $k\geq 0$, $x_k<x_{k+1}<x^*$ and $\lim_{k\to+\infty}x_k=x^*$.

IV. Example

In this section, we compare the proposed event-triggered and self-triggered communication protocols. We consider $N=6$ agents with the dynamics of a fourth-order oscillator where

$$A_r=\begin{bmatrix}0&-1&1&0\\1&0&0&1\\0&0&0&-\frac{1}{3}\\0&0&\frac{1}{3}&0\end{bmatrix}. \tag{44}$$

The agents exchange information according to the communication graph shown in Fig. 2a. Since $\sigma(A_r)\subset\mathrm{i}\mathbb{R}$, Assumption 1 is satisfied. Selecting all edge weights equal to $\rho=1$, we have that $\sigma(\mathcal{L})=\{0,0.5344,1.5\pm\mathrm{i}0.8660,2.2328\pm\mathrm{i}0.7926\}$, $\|\mathcal{L}\|=2.9364$, and $\beta=(1/3,1/3,1/3,0,0,0)$. The value of λ is obtained by finding $P\succeq I_{m(N-1)}$ and $\lambda>0$ such that $\hat{Z}^\top P+P\hat{Z}+2\lambda P\preceq 0$, where $\hat{Z}$ is defined in (21). Then, κ is as defined in the proof of Lemma 1 with

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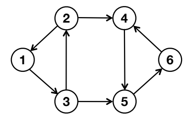
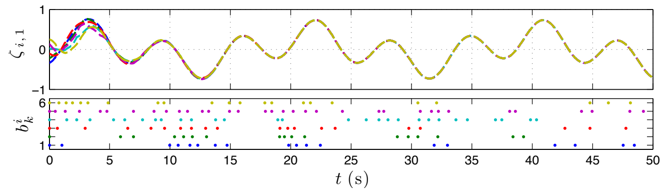
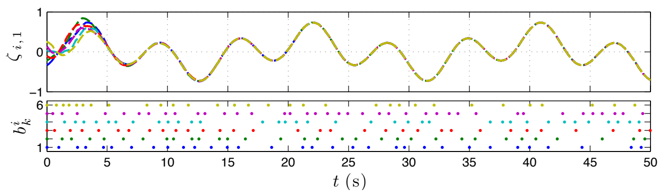


(a)



(b)



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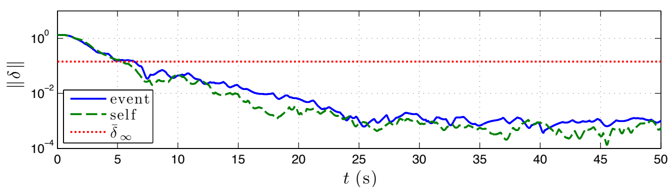

(d)

Fig. 2. Example. (a) Communication graph. (b)-(c) Trajectories of $\zeta_{i,1}$ and sequences of broadcast times for different communication protocols: (b) event-triggered; (c) self-triggered. (d) Trajectories of $\|\delta\|$ and its asymptotic bound.

$\kappa_1 = \sqrt{\lambda_{\max}(P)/\lambda_{\min}(P)}$. This yields $\lambda = 0.5344$ and $\kappa = 10.66$. The triggering parameters in (10) are $c_0 = 0.001$, $c_1 = 0.499$, and $\alpha = 0.25$. The agents initial conditions are $\zeta(0) = \hat{\zeta}(0) = g/\|g\|$ where $g \in \mathbb{R}^{mN}$ has entries $g_j = (2j - mN - 1)/(mN - 1)$ for $j \in \{1, \ldots, mN\}$. Note that $\|\zeta(0)\| = 1$ implies $\|\delta(0)\| \le \|I_N - \mathbf{1}_N \beta^\top\| = \sqrt{2}$. According to Theorem 2, we have $\bar{\delta} = 1.347 \times 10^2$, $\bar{v} = 3.990 \times 10^2$, $\bar{\delta}_\infty = 1.435 \times 10^{-1}$, and $\theta_{\min} = \theta_{\min}^{\text{self}} = 2.506 \times 10^{-6}$ s.

The simulation results are presented in Fig. 2b-d. The trajectories of $\zeta_{i,1}$ when using event-triggered and self-triggered communication protocols are shown in Fig. 2b and Fig. 2c, respectively (in the latter case, we solved (39) using two iterations of the method presented in Remark 1). In both cases, the difference between the trajectories of any two agents is within a certain error tolerance, a fact that is corroborated by the trajectory of $\|\delta\|$ shown in Fig. 2d. The average sampling intervals observed were between 1.4416 s and 2.8512 s in the event-triggered case and between 1.2654 s and 1.6235 s in the self-triggered case, illustrating the conservativeness introduced in the derivation of the latter communication protocol.

V. Conclusions

In this paper, we proposed and analyzed a control architecture designed to achieve synchronization of a multi-agent system using event-triggered and self-triggered communication protocols. The proposed event-triggered communication protocol extends the work reported in [18] for event-triggered consensus, by allowing directed communication graphs and more general agent dynamics. We showed that the proposed control architecture achieves bounded synchronization errors and that the closed-system does not exhibit Zeno solutions.

References

[1] W. Ren and R. W. Beard, *Distributed Consensus in Multi-vehicle Cooperative Control*, ser. Communications and Control Engineering. London: Springer-Verlag, 2008.

[2] R. Olfati-Saber and R. M. Murray, "Consensus problems in networks of agents with switching topology and time-delays," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1520–1533, Sep. 2004.

[3] J. Fax and R. Murray, "Information flow and cooperative control of vehicle formations," *IEEE Transactions on Automatic Control*, vol. 49, no. 9, pp. 1465–1476, Sep. 2004.

[4] L. Scardovi and R. Sepulchre, "Synchronization in networks of identical linear systems," *Automatica*, vol. 45, no. 11, pp. 2557–2562, Nov. 2009.

[5] P. Wieland, R. Sepulchre, and F. Allgöwer, "An internal model principle is necessary and sufficient for linear output synchronization," *Automatica*, vol. 47, no. 5, pp. 1068–1074, May 2011.

[6] P. Tabuada, "Event-triggered real-time scheduling of stabilizing control tasks," *IEEE Transactions on Automatic Control*, vol. 52, no. 9, pp. 1680–1685, Sep. 2007.

[7] K. J. Åström, "Event based control," in *Analysis and Design of Nonlinear Control Systems*, A. Astolfi and L. Marconi, Eds. Springer Berlin Heidelberg, 2008, pp. 127–147.

[8] J. Lunze and D. Lehmann, "A state-feedback approach to event-based control," *Automatica*, vol. 46, no. 1, pp. 211–215, Jan. 2010.

[9] M. C. F. Donkers and W. P. M. H. Heemels, "Output-based event-triggered control with guaranteed $\mathcal{L}_\infty$-gain and improved and decentralized event-triggering," *IEEE Transactions on Automatic Control*, vol. 57, no. 6, pp. 1362–1376, Jun. 2012.

[10] X. Wang and M. D. Lemmon, "Event-triggering in distributed networked control systems," *IEEE Transactions on Automatic Control*, vol. 56, no. 3, pp. 586–601, Mar. 2011.

[11] C. De Persis, R. Sailer, and F. Wirth, "On a small-gain approach to distributed event-triggered control," in *Preprints of the 18th IFAC World Congress*, Milano, Italy, 28 Aug. – 2 Sept. 2011, pp. 2401–2406.

[12] X. Wang, Y. Sun, and N. Hovakimyan, "Asynchronous task execution in networked control systems using decentralized event-triggering," *Systems & Control Letters*, vol. 61, no. 9, pp. 936–944, Sep. 2012.

[13] M. Velasco, P. Martí, and J. M. Fuertes, "The self triggered task model for real-time control systems," in *Proc. of the 24th IEEE Real-Time Systems Symp.*, Cancun, Mexico, 3–5 Dec. 2003, pp. 67–70.

[14] X. Wang and M. D. Lemmon, "Self-triggered feedback control systems with finite-gain $\mathcal{L}_2$ stability," *IEEE Transactions on Automatic Control*, vol. 54, no. 3, pp. 452–467, Mar. 2009.

[15] A. Anta and P. Tabuada, "To sample or not to sample: Self-triggered control for nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 55, no. 9, pp. 2030–2042, Sep. 2010.

[16] M. Mazo Jr., A. Anta, and P. Tabuada, "An ISS self-triggered implementation of linear controllers," *Automatica*, vol. 46, no. 8, pp. 1310–1314, Aug. 2010.

[17] D. V. Dimarogonas, E. Frazzoli, and K. H. Johansson, "Distributed event-triggered control for multi-agent systems," *IEEE Transactions on Automatic Control*, vol. 57, no. 5, pp. 1291–1297, May 2012.

[18] G. S. Seyboth, D. V. Dimarogonas, and K. H. Johansson, "Event-based broadcasting for multi-agent average consensus," *Automatica*, vol. 49, no. 1, pp. 245–252, Jan. 2013.

[19] G. Guo, L. Ding, and Q.-L. Han, "A distributed event-triggered transmission strategy for sampled-data consensus of multi-agent systems," *Automatica*, vol. 50, no. 5, pp. 1489–1496, May 2014.

[20] E. Garcia, Y. Cao, and D. W. Casbeer, "Decentralized event-triggered consensus with general linear dynamics," *Automatica*, vol. 50, no. 10, pp. 2633–2640, Oct. 2014.

[21] C. Godsil and G. Royle, *Algebraic Graph Theory*, ser. Graduate Texts in Mathematics. New York, USA: Springer-Verlag, 2001.

[22] R. Goebel, R. G. Sanfelice, and A. R. Teel, "Hybrid dynamical systems," *IEEE Control Systems Magazine*, vol. 29, no. 2, pp. 28–93, Apr. 2009.

[23] R. A. Horn and C. R. Johnson, *Topics in Matrix Analysis*. New York, NY, USA: Cambridge University Press, 1991.

[24] C. Van Loan, "The sensitivity of the matrix exponential," *SIAM J. on Numer. Analysis*, vol. 14, no. 6, pp. 971–981, Dec. 1977.