Self-triggered output feedback control of linear plants in the presence of unknown disturbances

João Almeida, Carlos Silvestre, and António M. Pascoal

Abstract—This note addresses the control of linear time-invariant plants in the presence of unknown, bounded disturbances when the output of the plant is only measured at sampling instants determined by a self-triggered strategy. In a self-triggered scenario, the controller is allowed to choose when the next sampling time should occur and does so based on available measurements and on a priori knowledge about the plant. The proposed solution is a cascade interconnection of a state observer and a self-triggered state feedback controller. We focus our attention on state observers that are only updated at sampling times and that are input-to-state stable (ISS) with respect to disturbances. Due to the cascade structure of the closed-loop system and the fact that self-triggered control strategies presented in the literature are ISS with respect to observation errors and exogenous disturbances, we conclude that the closed-loop is rendered ISS with respect to exogenous disturbances.

I. Introduction

The interaction between computers, networks, mobile devices, and physical systems has taken center stage in the area of robotics and has pushed the engineering community to consider new estimation and control paradigms that exploit to the fullest extent the new capabilities that are made available. Unlike the time driven approach of periodic control, today's engineers are looking into a more reactive approach where control actions are taken only when required, as determined by desired objectives. This new approach, commonly known as *event-triggered control*, is more suitable in applications where low energy consumption is sought and communications are costly or limited. In this context, as suggested by the name, control actions are driven by events generated by sensors, actuators, or users. For a concrete example, consider Fig. 1a.

In an event-triggered control scenario, an event detector is responsible for triggering a sampling action, typically whenever some function of the state or the output of the plant exceeds a prescribed threshold. Work on this subject may be found in [1]–[8]. The advantage of this approach versus a periodic sampling strategy is that the control input is only modified when some relevant change of the state or output of the plant occurs and this typically leads to a reduction in the number of samples required to achieve the same control objectives. Nonetheless, the state or output of the plant must be constantly monitored. To avoid this, *self-triggered control* strategies are proposed in [9]–[13] where,
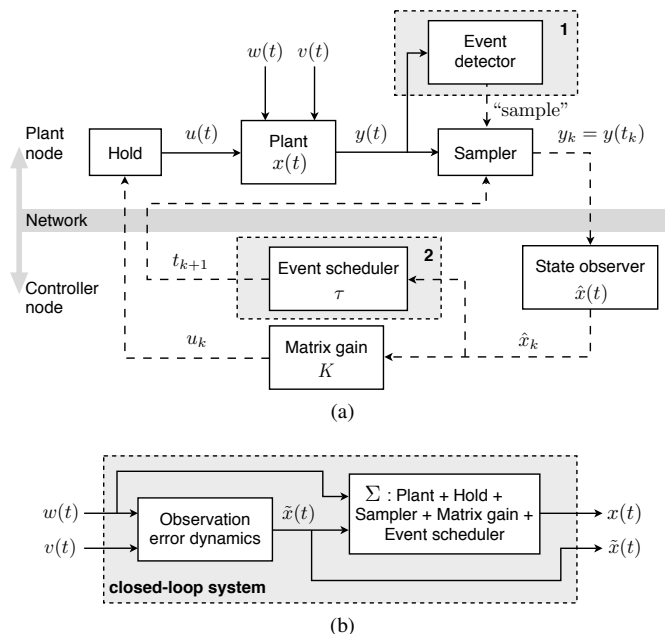
Fig. 1. (a) Output feedback control architecture: event-triggered (block **1** active, a sampling event is triggered when the sampler receives a "sample" message); self-triggered (block **2** active, a sampling event is triggered when the next sampling time t_{k+1} is reached). The state, the control input and the output of the plant are represented by x, u, and y, respectively, while w and v denote exogenous disturbances. Solid lines denote continuous time signals while dashed lines denote signals that are only updated at sampling times. (b) Diagram illustrating the cascade structure of the closed-loop system shown in (a) where $\tilde{x}$ denotes the observation error.

instead of continuously testing a triggering condition, an event scheduler is responsible for computing when the next sampling event should occur, based on the current sampled state or an estimate of it and on knowledge about the plant dynamics.

The above cited work on self-triggered control focuses solely on state feedback. In this note we widen the range of applicability of self-triggered control by introducing self-triggered *output feedback* control strategies. Namely, dynamic output feedback strategies where a state estimate computed by a state observer replaces the actual state in both the control law and the event scheduler. Related work on event-triggered output feedback control is reported in [4], [14]–[18] where a filter or observer is used to estimate the state of the plant which in turn is used to trigger sampling events. In most cases, a filter or observer is placed on the plant node and sometimes also on the controller node. Here, we introduce a structural difference by putting a state observer on the controller node only which is similar to the approach in [17] except that our observer is only updated at sampling times. It is our contention that it is better to place the observer on the controller node rather than

the plant node since the former is where greater computational resources are usually available. In [14], [18]–[20], the authors consider discrete-time plants in a stochastic setting and seek to optimize certain cost functions that may include penalty terms on the state and control inputs but also on the number of packets transmitted. To apply their results to a continuous time plant would require addressing discretization issues that are not discussed. In [16], [21], an alternative strategy that does not use observers is pursued where the triggering condition depends directly on the output of the plant. This approach is simpler to implement but some limitations are expected when compared to the observer based approach (analogous to the case of static versus dynamic output feedback control problems).

The goal of this note is to show that self-triggered output feedback control in the presence of unknown yet bounded disturbances is possible under the proposed control architecture, provided a suitable observer is available. By suitable observer, we mean a state observer that is only updated at sampling times and that can guarantee robustness to disturbances in an input-to-state stability[1] sense for any sequence of sampling times generated by the event scheduler. We show that if certain observability conditions are satisfied, there exists a sequence of gain matrices such that the proposed observer has the desired robustness requirements. Using the fact that scheduling methods proposed in the literature for the state feedback case (such as those in [10], [11], [13] to name a few) can be shown to be input-to-state stable (ISS) with respect to observation errors and exogenous disturbances, it is then straightforward to prove that the closed-loop system is ISS with respect to exogenous disturbances by exploiting the cascade structure of the closed-loop system (see Fig. 1b).

The note is organized as follows. In Section II, the proposed control architecture is introduced and the problem addressed is formally stated. In Section III, conditions that guarantee the existence of an ISS state observer are derived and one such observer is presented. In Section IV, an illustrative example with simulation results is provided. Finally, Section V contains some concluding remarks. To improve readability all proofs have been placed in the Appendix.

II. Self-triggered control

Consider a linear time-invariant plant with state $x \in \mathbb{R}^{n_x}$ and initial state $x(t_0) = x_0$ that satisfies, for all $t \geq t_0$,

$$\dot{x}(t) = Ax(t) + B_1 u(t) + B_2 w(t) \tag{1a}$$
$$y(t) = Cx(t) + Dv(t) \tag{1b}$$

where $u \in \mathbb{R}^{n_u}$ is the control input, $w \in \mathbb{R}^{n_w}$ and $v \in \mathbb{R}^{n_v}$ are exogenous disturbances, $y \in \mathbb{R}^{n_y}$ is the output of the plant and A, B_1, B_2, C, and D are matrices of appropriate dimensions. In the above we assume that the disturbances are bounded, that is, $\|w\|_{\mathcal{L}_\infty} < \infty$ and $\|v\|_{\mathcal{L}_\infty} < \infty$ where $\|x\|_{\mathcal{L}_\infty}$ is the $\mathcal{L}_\infty$-norm of a signal $x(t)$, defined as $\sup_{t \geq t_0} \|x(t)\|$ with $\|\cdot\|$ denoting the Euclidean norm. The pairs (A, B_1) and (A, C) are assumed to be controllable and observable, respectively.

[1]For the definition of input-to-state stability and related results the reader is referred to, e.g., [22, Chapter 4].

Our goal is to prove that the self-triggered output feedback control architecture proposed in Fig. 1a renders the closed-loop system ISS with respect to exogenous disturbances.

A. Control architecture

As depicted in the block diagram of Fig. 1a, the output of the plant y is sampled whenever $t = t_k$, where $\{t_k\}_{k \geq 1}$ denotes a sequence of sampling times. This information is then sent to the state observer on the controller node. The observer computes an estimate of the state of the plant at the current sampling time denoted by $\hat{x}_k$ and feeds this estimate to the matrix gain and the event scheduler. The control input is kept constant between sampling times in a zero-order hold manner, that is, for all $t \in [t_k, t_{k+1})$ and all $k \geq 0$,

$$u(t) = K\hat{x}_k. \tag{2}$$

The matrix gain K is such that $A + B_1 K$ is Hurwitz. Based on the current estimated state and on knowledge about the plant dynamics, the event scheduler computes when the next sampling time t_{k+1} should occur and communicates this information to the sampler. The computations performed by the scheduler are represented by a *scheduling function* $\tau : \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$ that maps states to time intervals such that

$$t_{k+1} - t_k = \tau_k = \tau(\hat{x}_k), \tag{3}$$

for all $k \geq 0$. The image of the function τ represents the set of possible sampling intervals.

The internal state of the state observer is denoted by $\hat{x}(t)$, a signal that is only used for analysis purposes as the actual implementation only requires the discrete state $\hat{x}_k = \hat{x}(t_k)$. We consider full-order state observers with an initial condition $\hat{x}_0 \in \mathbb{R}^{n_x}$ that satisfy, for all $k \geq 0$,

$$\hat{x}_{k+1}^- = F_k \hat{x}_k + G_k u_k \tag{4a}$$
$$\hat{x}_{k+1} = \hat{x}_{k+1}^- + H_{k+1}(y_{k+1} - C\hat{x}_{k+1}^-) \tag{4b}$$

where $F_k = F(\tau_k) = e^{A\tau_k}$, $G_k = \int_0^{\tau_k} e^{As}\mathrm{d}s B_1$, and $H_{k+1} \in \mathbb{R}^{n_x \times n_y}$ is a time-varying gain matrix to be defined. For $t \in [t_k, t_{k+1})$, the prediction step (4a) is the discrete time equivalent of the open loop dynamics

$$\dot{\hat{x}}(t) = A\hat{x}(t) + B_1 u_k, \tag{5}$$

where $\hat{x}_{k+1}^- = \hat{x}(t_{k+1}^-) = \lim_{\epsilon \to 0} \hat{x}(t_{k+1} - \epsilon)$. At time $t = t_{k+1}$, a new measurement y_{k+1} is received and $\hat{x}(t_{k+1})$ is updated according to (4b). Let the observation error associated with the state estimate $\hat{x}(t)$ be defined, for all $t \geq t_0$, as

$$\tilde{x}(t) = x(t) - \hat{x}(t). \tag{6}$$

According to (1a), (4b), and (5), the observation error satisfies

$$\begin{cases} \dot{\tilde{x}}(t) = A\tilde{x}(t) + B_2 w(t), & t \in [t_k, t_{k+1}) \\ \tilde{x}(t) = (I - H_{k+1}C)\tilde{x}(t^-) \\ \quad - H_{k+1}Dv(t), & t = t_{k+1} \end{cases} \tag{7}$$

for all $k \geq 0$. The observation error dynamics in (7) form a linear impulsive system, that is, a system with continuous time dynamics and discrete time updates or jumps.

We shall refer henceforth to the system that consists of the elements in Fig. 1a apart from the state observer as subsystem Σ. Using (1), (2), and (6), the dynamics of Σ may be written, for all $k \geq 0$, as

$$
\begin{cases}
\dot{x}(t) = (A + B_1 K)x(t) - B_1 K\hat{e}(t) \\
\qquad - B_1 K\tilde{x}(t) + B_2 w(t), & t \in [t_k, t_{k+1}) \\
x(t) = x(t^-), & t = t_{k+1}
\end{cases}
\quad (8)
$$

where the signals $\tilde{x}$ and w are regarded as exogenous disturbances, $\{t_k\}_{k \geq 1}$ is given by (3), and the input $\hat{e}(t) = \hat{x}(t) - \hat{x}_k$ for $t \in [t_k, t_{k+1})$ denotes the error induced by sampling.

The closed-loop system is thus described by (7) and (8) with augmented state $(x, \tilde{x})$ and disturbances w and v as inputs. An important feature of the proposed control architecture is the fact that, as shown in Fig. 1b, the closed-loop system is the cascade interconnection of the observation error dynamics in (7) and the subsystem Σ in (8). If the observation error dynamics is ISS with respect to the exogenous disturbances w and v, and subsystem Σ is ISS with respect to the observation error $\tilde{x}$ and the exogenous disturbance w, then the closed-loop system is ISS. This can be proved by resorting to the fact that a cascade interconnection of ISS systems is still ISS using arguments similar to the ones in [22, Lemma 4.7].

B. Main assumptions and problem formulation

The focus of this note is on how to select the sequence of observer gain matrices $\{H_k\}_{k \geq 1}$ such that the observation error dynamics in (7) is ISS. Therefore, we do not address how the matrix gain K or the scheduling function τ are designed. From here on, we assume that these elements are given and that the following assumption holds.

Assumption 1: The scheduling function τ is such that the subsystem Σ is ISS with respect to the observation error $\tilde{x}$ and the exogenous disturbance w.

It can be shown that Assumption 1 holds for the self-triggered state feedback controllers proposed in [10], [11] and [13] to name a few. To guarantee the existence of $\{H_k\}_{k \geq 1}$ such that (7) is ISS, we need to restrict the class of scheduling functions allowed.

Assumption 2: The image of the scheduling function τ is

$$
\mathcal{T}_{\tau_{\min}, \Delta, J} = \{\tau_{\min} + j\Delta : j = 0, 1, \ldots, J\} \quad (9)
$$

for some constant design parameters $\tau_{\min} > 0$, $\Delta > 0$, and $J \in \{0, 1, 2, \ldots\}$.

Assumption 2 asserts that the sequence of sampling intervals $\{\tau_k\}_{k \geq 0}$ has elements that belong to a finite set of equally spaced points, which may be written concisely as $\{\tau_k\}_{k \geq 0} \in \mathcal{T}_{\tau_{\min}, \Delta, J}$. By considering a finite set as in (9), we can perform an observability analysis in an attempt to identify possible problematic choices for the parameters $\tau_{\min}$ and Δ. If the image of τ were allowed to be an interval, the presence of a single pathological sampling interval would prevent us from claiming the existence of $\{H_k\}_{k \geq 1}$ such that (7) is ISS.

In light of the previous discussion, the problem at hand is formally stated next.

Problem 1: Find a sequence of observer gain matrices $\{H_k\}_{k \geq 1}$ such that the observation error dynamics in (7) is ISS

with respect to the exogenous disturbances w and v, *regardless* of the sequence of sampling intervals $\{\tau_k\}_{k \geq 0} \in \mathcal{T}_{\tau_{\min}, \Delta, J}$ generated by the event scheduler.

We will show that for an appropriate choice of $\tau_{\min}$ and Δ, there exists a sequence $\{H_k\}_{k \geq 1}$ that solves Problem 1.

III. STATE OBSERVER

The purpose of this section is to find a sequence of observer gain matrices $\{H_k\}_{k \geq 1}$ such that the observation error dynamics in (7) is ISS. Before discussing how to construct such a sequence, we first need to guarantee that such a sequence in fact exists. To this effect, we consider the discrete time equivalent of (1) in the absence of disturbances that satisfies

$$
\begin{aligned}
x_{k+1} &= F_k x_k + G_k u_k & (10a) \\
y_k &= C x_k & (10b)
\end{aligned}
$$

for all $k \geq 0$, where $x_k = x(t_k)$. We start this section with an analysis of the observability properties of (10) by resorting to linear systems theory. See, e.g., [23, Chapters 25 and 29] for an in-depth presentation of this subject. The first question that needs to be answered is whether the discrete time equivalent (12) is observable (in some well-defined sense). Although the continuous time plant (1) is assumed observable, sampling may cause a loss of observability for certain choices of $\tau_{\min}$ and Δ in (9) (see, e.g., [24, Chapter 3]). In what follows, we will show that for an appropriate choice of $\tau_{\min}$ and Δ the observability of the original continuous time-invariant plant (1) carries over to the discrete time-varying system (10).

A. Observability analysis

To analyze the observability properties of (10), we draw inspiration from the work reported in [25] where the authors give a sufficient condition for observability of a discretized switched linear system under arbitrary switching. Since τ_k can only take a finite number of values, only a finite number of matrices F_k are possible. Thus, system (10) can be viewed as a switched system with switching induced by the sequence of sampling intervals. The results in [25] cannot be applied directly to our case since in [25] the sampling period is fixed and the switching occurs in the output matrix of the plant. This is in contrast to our case, where the switching signal is the sequence of sampling intervals which induces a switching in the state matrix of the plant, while its output matrix is fixed. Thus, a modification of the results in [25] is required.

The work reported in [25] builds on the concept of van der Waerden numbers. Let $\mathbb{Z}$ denote the set of integers and $\mathbb{N}$ the set of positive integers. Given $a, b \in \mathbb{Z}$ such that $a \leq b$, let $\langle a, b \rangle$ denote the set $\{n \in \mathbb{Z} : a \leq n \leq b\}$. For $n, p \in \mathbb{N}$, the van der Waerden number $W(n, p)$ is the least $w \in \mathbb{N}$ such that any partition of $\langle 1, w \rangle$ into p parts has a part that contains a n-term arithmetic progression. The celebrated theorem of van der Waerden proves the existence of $W(n, p)$. For our results, we need to use a different, yet equivalent, formulation of the van der Waerden's Theorem borrowed from [26]. Let $G(n, m)$ denote the smallest $g \in \mathbb{N}$ such that if $\{a_i\}_{i=1}^g$ is a strictly increasing sequence of integers with gaps bounded by m (that is, $a_{i+1} - a_i \in \langle 1, m \rangle$ for all $i \in \langle 1, g - 1 \rangle$), then

$\{a_i\}_{i=1}^g$ contains a n-term arithmetic progression. The rate of progression for this arithmetic progression is between 1 and $R(n,m) = m \left(\left\lceil \frac{G(n,m)}{n-1} \right\rceil - 1 \right)$.

Before proceeding, we need to formalize our notion of observability. The observability matrix associated with (10) or with the pair (F_k, C) on an interval $\langle k_0, k_f \rangle$, with $k_f \geq k_0 + 1$, is defined as

$$
\mathcal{O}(k_0, k_f) = \begin{bmatrix} C \\ C\Phi(k_0 + 1, k_0) \\ C\Phi(k_0 + 2, k_0) \\ \vdots \\ C\Phi(k_f - 1, k_0) \end{bmatrix} \in \mathbb{R}^{(k_f - k_0)n_y \times n_x},
$$

where $\Phi(k, j)$ is the transition matrix associated with (10). The pair (F_k, C) is said to be *uniformly l-step observable* if there exists $l \in \mathbb{N}$ such that, for all $k \geq 0$, rank $\mathcal{O}(k, k+l) = n_x$. Given (A, C) observable, $T > 0$ is a *nonpathological sampling period* of A if (e^{AT}, C) is also observable.

Lemma 1: Suppose there exist $d_1, d_2 \in \mathbb{N}$ such that $d_1 \tau_{\min} = d_2 \Delta$. Let $\delta = \frac{\tau_{\min}}{d_2} = \frac{\Delta}{d_1}$ and $J^* = d_2 + d_1 J$. If, for all $r \in \langle 1, R(n_x, J^*) \rangle$, $r\delta$ is a nonpathological sampling period of A, then the pair (F_k, C) with $\{\tau_k\}_{k \geq 0} \in \mathcal{T}_{\tau_{\min}, \Delta, J}$ is uniformly $G(n_x, J^*)$-step observable. Moreover, the set of pathological values for δ is countable and the result holds for arbitrarily small δ.

Unfortunately, other than the trivial cases ($G(n, 1) = n$ and $G(2, m) = 2$), only a few values of $G(n, m)$ are known exactly ($G(3, \{2, 3, 4, 5, 6\}) = \{5, 9, 11, 17, 23\}, G(4, \{2, 3\}) = \{10, 26\}, G(5, 2) = 19, G(6, 2) = 37$). Some upper bounds are known for the remaining entries but they grow at an enormous rate with both n and m. This limits our ability to check if observability is preserved for large n_x or large J. Nevertheless, almost all values of $\tau_{\min}$ and Δ are nonpathological.

Consider the following well-known sufficient condition for identifying pathological sampling periods (see, e.g., [25, Theorem 1]). Let $\sigma(A)$ denote the spectrum of matrix A (the set of all eigenvalues of A). Also, let $\Re\{z\}$ and $\Im\{z\}$ denote the real and imaginary parts of a complex number z, respectively. A given $T > 0$ is nonpathological if, for all $\lambda, \mu \in \sigma(A)$ and all $q \in \mathbb{N}$, $|\Im\{\lambda - \mu\}| \neq \frac{2\pi q}{T}$. Let

$$
\mathcal{F}(A) = \left\{ \frac{|\Im\{\lambda - \mu\}|}{2\pi} : \lambda, \mu \in \sigma(A), \lambda \neq \mu, \Re\{\lambda\} = \Re\{\mu\} \right\}.
$$

Resorting to Lemma 1, we have that the set of pathological sampling periods is a subset of the countable set $\left\{ \frac{q}{rf} : q \in \mathbb{N}, f \in \mathcal{F}(A), r \in \langle 1, R(n_x, J^*) \rangle \right\}$. We can further derive a conservative yet simpler sufficient condition that guarantees the preservation of observability. Let $\mathbb{Q}$ denote the set of rational numbers. If, for all $f \in \mathcal{F}(A)$, we have that $\frac{f}{f_0} \notin \mathbb{Q}$ where $f_0 = \frac{1}{\delta}$ (in which case f and f_0 are said to be *irrationally related*), then $r\delta$ is nonpathological for all $r \in \mathbb{N}$.

B. ISS state observer

Having shown how observability can be preserved by avoiding certain values for the parameters $\tau_{\min}$ and Δ, we now address how to select the sequence of observer gain matrices such that Problem 1 is solved. We start by noting

that, in the absence of disturbances, (10) and (4) imply that the observation error at sampling times satisfies, for all $k \geq 0$,

$$
\tilde{x}_{k+1} = (I - H_{k+1}C)F_k \tilde{x}_k. \tag{11}
$$

Assume that $\{H_k\}_{k \geq 1}$ is uniformly bounded and note that $\{\tau_k\}_{k \geq 0}$ satisfies $0 < \tau_{\min} \leq \tau_k \leq \tau_{\max} = \tau_{\min} + \Delta J$, for all $k \geq 0$. If $\{H_k\}_{k \geq 1}$ is such that the discrete time system (11) is globally uniformly exponentially stable (GUES), then it can be shown that the continuous time system (7) is ISS. Thus, Problem 1 is solved if we can find a sequence $\{H_k\}_{k \geq 1}$ such that (11) is GUES for all sequences of sampling intervals $\{\tau_k\}_{k \geq 0} \in \mathcal{T}_{\tau_{\min}, \Delta, J}$ and for all initial conditions $\tilde{x}_0 \in \mathbb{R}^{n_x}$.

To define one such sequence of gain matrices, we first need to strengthen the observability properties of (10). To accomplish this, we resort to Gramian matrices, an alternate way of characterizing observability. The system (10) is observable on $\langle k_0, k_f \rangle$ if and only if the *observability Gramian*

$$
M_{\mathcal{O}}(k_0, k_f) = \sum_{j=k_0}^{k_f - 1} \Phi^\top(j, k_0) C^\top C \Phi(j, k_0)
$$

is positive definite. If there exist $l \in \mathbb{N}$ and $\varepsilon_1, \varepsilon_2 > 0$ such that, for all $k \geq 0$,

$$
\varepsilon_1 I \preceq M_{\mathcal{O}}(k - l + 1, k + 1) \preceq \varepsilon_2 I,
$$

then (10) is said to be *uniformly completely observable* (UCO).

The analysis performed in Section III-A, showed that (10) is uniformly l-step observable under some conditions on parameters $\tau_{\min}$ and Δ. In general, uniform l-step observability does not imply uniform complete observability. However, in our case, since $\mathcal{T}_{\tau_{\min}, \Delta, J}$ is a compact set, this is true.

Lemma 2: If the plant (10) with $\{\tau_k\}_{k \geq 0} \in \mathcal{T}_{\tau_{\min}, \Delta, J}$ is uniformly l-step observable, then it is UCO.

A complementary notion to observability is that of reconstructibility (sometimes simply constructibility). The *reconstructibility Gramian* is defined as

$$
M_{\mathcal{R}}(k_0, k_f) = \sum_{j=k_0}^{k_f - 1} \Phi^\top(j, k_f) C^\top C \Phi(j, k_f). \tag{12}
$$

The definition of *uniformly completely reconstructible* (UCR) is similar to the definition of UCO.

Theorem 1: Given $\eta > 1$, if the system (10) is UCR, then the state observer (4) with time-varying gain matrix given by

$$
H_{k+1} = [M_{\mathcal{R}, \eta}(k - l + 1, k + 1)]^{-1} C^\top \tag{13}
$$

for all $k \geq 0$, where

$$
M_{\mathcal{R}, \eta}(k_0, k_f) = \sum_{j=k_0}^{k_f - 1} \eta^{4(j - k_f + 1)} \Phi^\top(j, k_f) C^\top C \Phi(j, k_f),
$$

is GUES with a decay rate of η for all $\{\tau_k\}_{k \geq 0} \in \mathcal{T}_{\tau_{\min}, \Delta, J}$ and all $\tilde{x}_0 \in \mathbb{R}^{n_x}$. Moreover, $\{H_k\}_{k \geq 1}$ is uniformly bounded.

Theorem 1 may be proven following the same arguments used in [23, Theorem 29.2, Note 29.2]. To satisfy the hypothesis of Theorem 1, one has to show the following.

Lemma 3: Under the assumptions of Lemma 1, the system (10) is UCR.

Lemma 3 and Theorem 1 imply that (11) is GUES when $\{H_k\}_{k \geq 1}$ is given by (13). Thus, Problem 1 is solved.

IV. AN ILLUSTRATIVE EXAMPLE

We now illustrate the previous results with a third order linear plant, modeled as in (1), where

$$A = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & -2 & 2 \end{bmatrix}, B_1 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}, B_2 = I_3, C = \begin{bmatrix} 1 \\ 0 \\ 0 \end{bmatrix}^\top, D = 1.$$

Throughout the example, time is expressed in seconds. The pairs (A, B_1) and (A, C) are controllable and observable, respectively, the spectrum of A is $\sigma(A) = \{1, \frac{1}{2} \pm i\frac{\sqrt{3}}{2}\}$, and $\mathcal{F}(A) = \{\frac{\sqrt{3}}{2\pi}\}$. For demonstration purposes, we consider the scheduling method presented in [10]. Taking $\gamma = 100$ in [10, Eq. (2)], we obtain the gain matrix $K = [-2.4158 \ -0.8955 \ -4.6065]$. The scheduling function is defined as $\tau(x) = \frac{1}{\rho} \ln\left(1 + \frac{\rho x^\top N_2 x}{x^\top A_{cl}^\top N_1 A_{cl} x}\right)$, where $\beta = 0.9$, $N_1 = (1 - \beta^2)I + K^\top K$, $N_2 = \frac{1-\beta^2}{2}I + K^\top K$, $A_{cl} = A + B_1 K$, and $\rho = 15.0491$. Since the image of the function τ is an interval of the form $[\tau_{\min}^*, \tau_{\max}^*]$, we force its image to be $\mathcal{T}_{\tau_{\min}, \Delta, J}$ by defining a new scheduling function $\tau_{\text{grid}} : \mathbb{R}^{n_x} \to \mathbb{R}_{\geq 0}$ as $\tau_{\text{grid}}(x) = \tau_{\min} + \min\left\{\left\lfloor \frac{\tau(x) - \tau_{\min}}{\Delta} \right\rfloor, J\right\} \Delta$, where the design parameters satisfy $0 < \tau_{\min} \leq \tau_{\min}^*$, $\Delta > 0$, and $J \leq \lfloor (\tau_{\max}^* - \tau_{\min}^*)/\Delta \rfloor$. The stability properties that hold when τ is used, also hold for τ_{grid} since $\tau_{\text{grid}}(x) \leq \tau(x)$ for all $x \in \mathbb{R}^{n_x}$. The maximum and minimum sampling intervals allowed are set to $\tau_{\min} = 0.015$ and $\tau_{\max} = 0.285$, respectively, and the step size is $\Delta = 3\tau_{\min} = 0.045$ ($d_1 = 3$, $d_2 = 1$, $J = 6$). It can be verified that the values of $\tau_{\min}$ and Δ are nonpathological using the irrationally related argument. Since the value of $G(3, 19)$ is not known and is possibly quite large, we computed the index of observability by testing all possible sequences of sampling intervals and found the system to be 3-step observable. However, we decided to use a larger window size of 7 to ensure good numerical properties of the state observer. The decay rate of the observer is set to $\eta = 2$.

The closed-loop system is simulated on the time interval $[0, 50]$. The disturbances w and v are zero for $t \in [0, 25)$ and for $t \in [25, 50]$ they are such that $\|w\|_{\mathcal{L}_\infty} = 10$ and $\|v\|_{\mathcal{L}_\infty} = 10$. The plant and the observer are initialized with $x_0 = [-1 \ 2 \ -1]^\top$ and $\hat{x}_0 = [0 \ 0 \ 0]^\top$, respectively. Fig. 2 shows the evolution of the plant state norm and of the observation error norm at sampling times, and also the sequence of sampling intervals generated by the event scheduler. A total of 532 samples are taken resulting in an average sampling interval of 0.1054 for $t \in [0, 25)$ and 0.0850 for $t \in [25, 50]$. Both the state of the plant and the observation error tend to zero asymptotically in the absence of disturbances and stay bounded in the presence of bounded disturbances.

V. CONCLUSIONS

This note addressed the control of linear plants in the presence of unknown disturbances when the output is sampled using a self-triggering strategy. The proposed solution builds on previous results on self-triggered state feedback control and uses an observer based approach to extend them to the dynamic output feedback case. We have shown that for an appropriate choice of some design parameters, the proposed
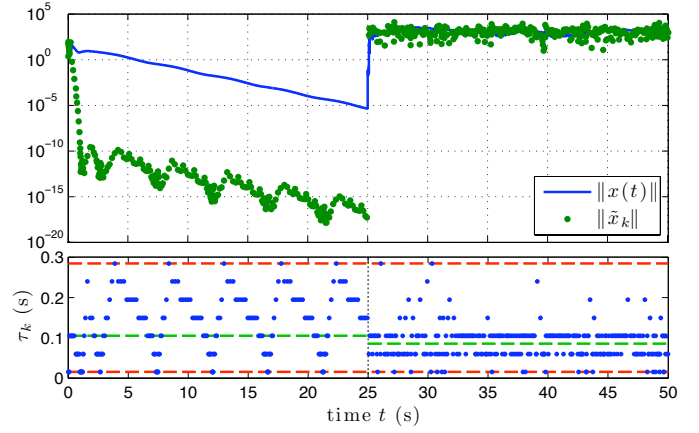

Fig. 2. Simulation results of the closed-loop system using the scheduling method of [10]. Top plot: time evolution of the plant state norm and of the observation error norm at sampling times. Bottom plot: sequence of sampling intervals (dashed green line indicates the average sampling interval).

observer is ISS with respect to exogenous disturbances, regardless of the sequence of sampling intervals generated, thereby concluding that the same holds for the closed-loop system. An illustrative example with simulation results demonstrated the steps required to apply the proposed controller.

APPENDIX

Proof of Lemma 1: For all $k \geq 0$, we have that $\tau_k = \tau_{\min} + s_k \Delta$ with $s_k \in \langle 0, J \rangle$ because of the structure of (9). By hypothesis, we have that $\tau_{\min} = d_2 \delta$ and $\Delta = d_1 \delta$. Hence, we may write $\tau_k = p_k \delta$ with $p_k \in \langle 1, J^* \rangle$. Letting $E = e^{\delta A^\top}$, the matrix F_k in (10) can be written as $F_k = e^{A p_k \delta} = (E^{p_k})^\top$. Suppose $k \geq 0$ is fixed and $l \geq 1$ is given. Then, the observability matrix associated with (10) on $\langle k, k+l \rangle$ satisfies

$$\mathcal{O}^\top(k, k+l) = \begin{bmatrix} E^{q_1} C^\top & E^{q_2} C^\top & \cdots & E^{q_l} C^\top \end{bmatrix}, \quad (14)$$

where $\{q_i\}_{i=1}^l$ denotes the sequence formed by the cumulative sum of $\{p_i\}_{i=k}^{k+l-2}$, that is, $q_1 = 0$, $q_2 = p_k$, $q_3 = p_k + p_{k+1}$, ..., $q_l = \sum_{i=0}^{l-2} p_{k+i}$. Notice that $q_{i+1} - q_i = p_{k+i-1} \in \langle 1, J^* \rangle$ for all $i \in \langle 1, l-1 \rangle$. If we choose $l \geq G(n_x, J^*)$, then $\{q_i\}_{i=1}^l$ will contain at least one arithmetic progression of length n_x. Select one such progression and let $i_1 \in \langle 1, l - n_x + 1 \rangle$ denote the index of its first term. Then, for all $j \in \langle 1, n_x \rangle$, we have that $q_{i_j} = q_{i_1} + r(j-1)$ for some rate of progression $r \in \langle 1, R(n_x, J^*) \rangle$. This implies that the matrix $\mathcal{O}(k, k+l)$ given in (14) contains the submatrix

$$\begin{bmatrix} E^{q_{i_1}} C^\top & E^{q_{i_2}} C^\top & \cdots & E^{q_{i_{n_x}}} C^\top \end{bmatrix}$$
$$= E^{q_{i_1}} \begin{bmatrix} C^\top & E^r C^\top & \cdots & (E^r)^{n_x - 1} C^\top \end{bmatrix} = E^{q_{i_1}} \widehat{\mathcal{O}}_r^\top. \quad (15)$$

The matrix $\widehat{\mathcal{O}}_r$ in (15) is the observability matrix associated with the pair (E^r, C). Since E is invertible, and therefore also $E^{q_{i_1}}$, if $\text{rank} \widehat{\mathcal{O}}_r = n_x$ for all $r \in \langle 1, R(n_x, J^*) \rangle$, then $\text{rank} \mathcal{O}(k, k+l) = n_x$, that is, the system is observable on $\langle k, k+l \rangle$. This condition is equivalent to the hypothesis of the lemma. Because the choice of l does not depend on k, the same conclusion is valid for all $k \geq 0$, implying that the system is uniformly $G(n_x, J^*)$-step observable.

Following arguments similar to those in [27, Proof of Lemma 1], there exists a nonsingular matrix S such that

$$(S\widehat{\mathcal{O}}_r)^\top = \begin{bmatrix} C^\top & \frac{E^r - I}{r\delta}C^\top & \cdots & \left(\frac{E^r-I}{r\delta}\right)^{n_x-1}C^\top \end{bmatrix}. \quad (16)$$

Since the determinant of (16) is a continuous function of δ, we may take its limit as δ tends to zero, which yields the determinant of

$$\begin{bmatrix} C^\top & (CA)^\top & \cdots & (CA^{n_x-1})^\top \end{bmatrix}. \quad (17)$$

By assumption, the determinant of (17) is nonzero and therefore the determinant of (16) is nonzero for sufficiently small δ. Moreover, since the determinant of (16) is an analytical function of δ for all $r \in \langle 1, R(n_x, J^*) \rangle$ that is not identically zero, its set of zeros must be countable. ∎

Proof of Lemma 2: Since F_k is uniformly bounded for all $k \geq 0$, the existence of ε_2 is guaranteed. To show that ε_1 is positive, we use the fact that the observability Gramian satisfies $M_{\mathcal{O}}(a,b) = \mathcal{O}^\top(a,b)\mathcal{O}(a,b)$, yielding

$$\varepsilon_1 = \inf_{k\geq 0} \lambda_{\min}\{M_{\mathcal{O}}(k, k+l)\}$$
$$= \inf_{k\geq 0} \lambda_{\min}\{\mathcal{O}^\top(k, k+l)\mathcal{O}(k, k+l)\}.$$

Note that the entries of the matrix $\mathcal{O}(k, k+l)$ are continuous functions of $\tau_k, \ldots, \tau_{k+l-2}$. Therefore, the function $f_l : \mathbb{R}^{l-1} \to \mathbb{R}$ defined as

$$f_l(\tau_k, \ldots, \tau_{k+l-2}) = \lambda_{\min}\{\mathcal{O}^\top(k, k+l)\mathcal{O}(k, k+l)\},$$

is also a continuous function of $\tau_k, \ldots, \tau_{k+l-2}$. Since $\mathcal{T}_{\tau_{\min},\Delta,J}$ is a compact set and therefore also the Cartesian product $\mathcal{T}_{\tau_{\min},\Delta,J}^{l-1}$, we have that the function f_l attains a maximum and a minimum, that is, its image is a closed interval. By hypothesis, each element of this interval is positive. Therefore, we have that $\varepsilon_1 = \min f_l > 0$ and the result follows. ∎

Proof of Lemma 3: Let $\phi_1 = \inf_{k\geq 0} \lambda_{\min}\{\Phi^\top(k - l + 1, k+1)\Phi(k-l+1, k+1)\}$ and $\phi_2 = \sup_{k\geq 0} \lambda_{\max}\{\Phi^\top(k-l+1, k+1)\Phi(k-l+1, k+1)\}$. Since $\bar{F}(\tau)$ is bounded for all $\tau \in \mathcal{T}_{\tau_{\min},\Delta,J}$, $\Phi(k-l+1, k+1)$ is bounded for all $k \geq 0$. Hence, the existence of ϕ_2 is guaranteed. To show that ϕ_1 is positive, note that $\Phi(k-l+1, k+1)$ is a continuous function of $\tau_{k-l+1}, \ldots, \tau_k$ and that it is nonsingular for all $k \geq 0$, since it is the product of matrices of the form $F(\tau)$ that are nonsingular for all $\tau \in \mathbb{R}$. Resorting to arguments similar to those used in the proof of Lemma 2, we conclude that $0 < \phi_1 \leq \phi_2 < +\infty$.

The hypothesis of the lemma together with Lemma 2, imply that (10) is UCO. Therefore, there exist $l \in \mathbb{N}$ and $\varepsilon_1, \varepsilon_2 > 0$ such that, for all $k \geq 0$, $\varepsilon_1 I \preceq M_{\mathcal{O}}(k - l + 1, k+1) \preceq \varepsilon_2 I$. Using the fact that $M_{\mathcal{R}}(a,b) = \Phi^\top(a,b)M_{\mathcal{O}}(a,b)\Phi(a,b)$, yields $\delta_1 I \preceq M_{\mathcal{R}}(k - l + 1, k+1) \preceq \delta_2 I$ where $\delta_i = \phi_i \varepsilon_i$ for $i = 1, 2$. This shows that (10) is UCR. ∎

REFERENCES

[1] K.-E. Årzén, "A simple event-based PID controller," in *Preprints of the 14th IFAC World Congress*, Beijing, P.R. China, Jan. 1999, pp. 423–428.

[2] K. J. Yook, D. M. Tilbury, and N. R. Soparkar, "Trading computation for bandwidth: Reducing communication in distributed control systems using state estimators," *IEEE Transactions on Automatic Control*, vol. 10, no. 4, pp. 503–518, Jul. 2002.

[3] P. Tabuada, "Event-triggered real-time scheduling of stabilizing control tasks," *IEEE Transactions on Automatic Control*, vol. 52, no. 9, pp. 1680–1685, Sep. 2007.

[4] K. J. Åström, "Event based control," in *Analysis and Design of Nonlinear Control Systems*, A. Astolfi and L. Marconi, Eds. Springer Berlin Heidelberg, 2008, pp. 127–147.

[5] W. P. M. H. Heemels, J. H. Sandee, and P. P. J. van den Bosch, "Analysis of event-driven controllers for linear systems," *Int. Journal of Control*, vol. 81, no. 4, pp. 571–590, Apr. 2008.

[6] M. Rabi, K. H. Johansson, and M. Johansson, "Optimal stopping for event-triggered sensing and actuation," in *Proc. of the 47th Conf. on Decision and Control*, Cancun, Mexico, Dec. 2008, pp. 3607–3612.

[7] J. Lunze and D. Lehmann, "A state-feedback approach to event-based control," *Automatica*, vol. 46, no. 1, pp. 211–215, Jan. 2010.

[8] M. Lemmon, "Event-triggered feedback in control, estimation, and optimization," in *Networked Control Systems*, ser. Lecture Notes in Control and Information Sciences, A. Bemporad, M. Heemels, and M. Johansson, Eds. Springer London, 2011, vol. 406, pp. 293–358.

[9] M. Velasco, P. Martí, and J. M. Fuertes, "The self triggered task model for real-time control systems," in *Proc. of the 24th IEEE Real-Time Systems Symp.*, Cancun, Mexico, 3–5 Dec. 2003, pp. 67–70.

[10] X. Wang and M. D. Lemmon, "Self-triggered feedback control systems with finite-gain $\mathcal{L}_2$ stability," *IEEE Transactions on Automatic Control*, vol. 54, no. 3, pp. 452–467, Mar. 2009.

[11] ——, "Self-triggering under state-independent disturbances," *IEEE Transactions on Automatic Control*, vol. 55, no. 6, pp. 1494–1500, Jun. 2010.

[12] A. Anta and P. Tabuada, "To sample or not to sample: Self-triggered control for nonlinear systems," *IEEE Transactions on Automatic Control*, vol. 55, no. 9, pp. 2030–2042, Sep. 2010.

[13] M. Mazo Jr., A. Anta, and P. Tabuada, "An ISS self-triggered implementation of linear controllers," *Automatica*, vol. 46, no. 8, pp. 1310–1314, Aug. 2010.

[14] L. Li and M. Lemmon, "Event-triggered output feedback control of finite horizon discrete-time multi-dimensional linear processes," in *Proc. of the 49th Conf. on Decision and Control*, Atlanta, GA, USA, Dec. 2010, pp. 3221–3226.

[15] D. Lehmann and J. Lunze, "Event-based output-feedback control," in *Proc. of the 19th Mediterranean Conf. on Control and Autom.*, Corfu, Greece, Jun. 2011, pp. 982–987.

[16] S. Trimpe and R. D'Andrea, "An experimental demonstration of a distributed and event-based state estimation algorithm," in *Preprints of the 18th IFAC World Congress*, Milano, Italy, Aug. 2011, pp. 8811–8818.

[17] M. C. F. Donkers and W. P. M. H. Heemels, "Output-based event-triggered control with guaranteed $\mathcal{L}_\infty$-gain and improved and decentralized event-triggering," *IEEE Transactions on Automatic Control*, vol. 57, no. 6, pp. 1362–1376, Jun. 2012.

[18] A. Molin and S. Hirche, "On the optimality of certainty equivalence for event-triggered control systems," *IEEE Transactions on Automatic Control*, vol. 58, no. 2, pp. 470–474, Feb. 2013.

[19] Y. Xu and J. Hespanha, "Communication logic design and analysis for networked control systems," in *Current Trends in Nonlinear Systems and Control*, ser. Systems and Control: Foundations & Applications, L. Menini, L. Zaccarian, and C. Abdallah, Eds. Birkhäuser Boston, 2006, pp. 495–514.

[20] R. Cogill, "Event-based control using quadratic approximate value functions," in *Proc. of the 48th Conf. on Decision and Control*, Shanghai, P.R. China, Dec. 2009, pp. 5883–5888.

[21] H. Yu and P. J. Antsaklis, "Event-triggered real-time scheduling for stabilization of passive and output feedback passive systems," in *Proc. of the 2011 American Control Conf.*, San Francisco, CA, USA, Jun. 2011, pp. 1674–1679.

[22] H. K. Khalil, *Nonlinear Systems*, 3rd ed. Upper Saddle River, New Jersey, USA: Prentice Hall, 2002.

[23] W. J. Rugh, *Linear System Theory*, 2nd ed. Upper Saddle River, NJ, USA: Prentice-Hall, 1996.

[24] T. Chen and B. A. Francis, *Optimal Sampled-Data Control Systems*. Berlin, Germany: Springer-Verlag, 1995.

[25] M. Babaali and M. Egerstedt, "Nonpathological sampling of switched linear systems," *IEEE Transactions on Automatic Control*, vol. 50, no. 12, pp. 2102–2105, Dec. 2005.

[26] J. R. Rabung, "On applications of van der Waerden's theorem," *Mathematics Magazine*, vol. 48, pp. 142–148, 1975.

[27] T. Hagiwara and M. Araki, "Design of a stable state feedback controller based on the multirate sampling of the plant output," *IEEE Transactions on Automatic Control*, vol. 33, no. 9, pp. 812–819, Sep. 1988.