Continuous-time consensus with discrete-time communications[☆]

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Abstract

This paper addresses the problem of reaching consensus among a group of agents that evolve in continuous-time and exchange information at discrete-time instants, referred to as update times. Each agent has its own sequence of update times and therefore the agents are not required to keep synchronized clocks among them. At each update time, an agent receives from a subset of the other agents their state, as determined by the communication topology that may be time-varying. Due to transmission delays, the information may be received by an agent with latency. In our proposed solution, the state of each agent is augmented with an extra state variable that is updated instantaneously at update times. Between updates, the original state and the extra variable both evolve in a continuous fashion. It is shown that consensus is reached asymptotically by reducing the original problem involving continuous-time variables and asynchronous communications to a discrete-time equivalent and using known results for discrete-time consensus.

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1. Introduction

The analysis of how groups of agents may exhibit complex global behaviors that emerge from simple local rules is a topic of research that has attracted considerable attention in recent years in many scientific fields that range from biology to space exploration. Among a multitude of problems that arise in this context, one that is easy to formulate and of practical interest is the consensus or agreement problem. This problem usually arises when a group of agents perform a task that requires coordination among them such as in rendezvous maneuvers, formation control of autonomous vehicles [1, 2], and distributed parallel computation over computer networks [3]. A fundamental part of such tasks relies on the agents being able to agree on some common knowledge that is not available a priori and that may need to be recomputed over time upon request or when the operating conditions change. Sharing of information through some form of communication channel is thus crucial to the successful completion of the task at hand and brings about such issues as switching network topologies and transmission delays.

Conditions under which consensus is reached asymptotically have been studied extensively in the literature by adopting different mathematical setups. In some cases, the state variables of each agent evolve in continuous-time (the state trajectory is a continuous curve) [4, 5, 6, 7, 8, 9], while in other cases they evolve in discrete-time (the state trajectory is a sequence of values) [10, 4, 5, 7, 11, 12, 13, 14, 15, 16, 17, 18]. Also relevant is whether the communication topology is fixed over time or if it is allowed to change, that is, if the information available to a given agent always comes from the same subset of agents or if this subset is allowed to change over time. Most of the literature has focused on the second, more challenging, case (in the list of references provided above, only [4] considers fixed topologies). Because the network topology is time-varying it may happen that it is disconnected so often that consensus is impossible to reach. Therefore, some connectivity assumptions on the sequence of graph topologies have to be made. Another issue that has to be taken into account are transmission delays. Since each agent has to send and receive data, consensus protocols will have to work with outdated information and thus be robust against time delays. Such issues are addressed in [10, 4, 6, 11, 12, 13, 8, 14, 15, 16, 18, 17, 9]. The above mentioned references cover thoroughly the case when the agents have first order dynamics. Results for higher-order dynamics may be found in [1, 2, 19, 20, 21, 22] and references therein.

In this paper, we consider agents with first order dynamics, that is, each agent's state is represented by a scalar. While some applications may require continuous state variables, continuous communication links among agents are hard to achieve in practice, where transmission of data naturally occurs in small bursts over short periods of time. In this paper, we take this communication constraint into account and consider a setup where the agents' state variables are continuous but information from neighboring agents is only available at discrete time instants that will henceforth be referred to as *update times*. Each agent has its own sequence of update times and therefore the agents are not required to keep synchronized clocks among them. Furthermore, these update times are not assumed to be uniformly spaced in time.

In our proposed solution, an extra state variable is introduced for each agent that evolves in continuous-time and is allowed to have discontinuities, unlike the primary state variable that is continuous for all time. Between update times, both the original state and the extra variable evolve continuously according to some specified dynamics. At update times, the original state variable keep their current values, while the extra variable is updated by forming a convex combination of state values received from other agents. To analyse the resulting system, we start by constructing a discrete-time equivalent of the continuous-time system. We then prove that the latter reaches consensus asymptotically, in the presence of switching topologies and time delays, by resorting to a well known results on discrete-time consensus.

It is interesting to remark that in [16] (see also [15]), a solution is proposed for a problem similar to the one addressed in this paper that also involves augmenting the state of each agent with an additional state variable, referred to as a way-point. Between update times, the original state of each agent changes continuously from its current value to the corresponding way-point, as determined by a pre-specified continuous function. At update times, the way-points are updated according to a averaging protocol with fixed weights. In our setup, for greater flexibility in the design of the consensus control law, we allow for time-varying weights; we further address explicitly the situation where the information received by one agent from the other agents may be outdated due to transmission delays (not considered in [16]).

An alternative strategy that does not require extra state variables is proposed in [23], where the authors consider a setup identical to ours (time-varying weights in the discrete updates, switching topologies, and time delays). Compared with [23], as noted above, our strategy has extra degrees of freedom in the form of weights in the continuous dynamics that yield further controller tuning parameters (see Remark 1 in Section 3.1 for further details). Furthermore, our approach exploits an interesting connection between a continuous-time consensus problem with discrete-time asynchronous communications and a related purely discrete-time one.

The paper is organized as follows. In Section 2, we begin by presenting a brief summary of basic concepts from graph theory that are important to understand the modeling of an inter-agent communication graph and its properties. The discrete-time consensus problem is then introduced and conditions required for asymptotic consensus are provided. In Section 3, a formal statement of the problem addressed in this paper is given along with a description of our proposed solution, related assumptions, and the convergence analysis. The main result of the paper asserts that our solution reaches consensus asymptotically under the stated assumptions. In Section 4, an illustrative example with numerical simulations is presented. Finally, concluding remarks are given in Section 5.

2. Discrete-time consensus

In this section we introduce a well-known discrete-time consensus problem and a related convergence result that will be crucial in proving convergence of our proposed consensus algorithm later, in Section 3. We begin by reviewing some key concepts from graph theory that play an important role in what follows. See [24] for an in-depth presentation of this subject.

2.1. Directed graphs

A directed graph or digraph $\mathcal{G} = \mathcal{G}(V, E)$ consists of a finite set $V = \{1, 2, \ldots, n\}$ of n vertices and a finite set $E \subseteq V \times V$ of m ordered pairs of vertices (i, j) named arcs. If (i, j) belongs to E then we say that i is adjacent to j. A path in $\mathcal{G}$ from i to j is a sequence of distinct vertices starting with i and ending with j such that consecutive vertices are adjacent. A vertex i is a root if there exists a path in $\mathcal{G}$ from vertex i to every other vertex in $\mathcal{G}$. If a graph has at least one root, we say that it is a rooted graph. Given a collection of graphs $\{\mathcal{G}_k = (V, E_k)\}_{k=1}^{B}$ of length B with the same vertex set, the union of these graphs (union graph) is defined as

$$\bigcup_{k=1}^{B} \mathcal{G}_k = \mathcal{G}_1 \cup \mathcal{G}_2 \cup \cdots \cup \mathcal{G}_B = (V, E_1 \cup E_2 \cup \cdots \cup E_B). \tag{1}$$

2.2. Problem formulation and known results

Consider a set of N agents, labeled 1 through N, each one with its own scalar state variable $x_i \in \mathbb{R}$. Each agent updates its state variable according to the equation

$$x_i(k+1) = \sum_{j=1}^{N} a_{ij}(k) x_j(k - \tau_{ij}(k)), \tag{2}$$

where $a_{ij}(k)$ are nonnegative coefficients, and $\tau_{ij}(k)$ are nonnegative integer time delays. Let $A(k)$ denote the matrix whose entries are $a_{ij}(k)$. Given $p, q \in \mathbb{Z}$ such that $p \leq q$, let $\langle p, q \rangle = \{m \in \mathbb{Z} : p \leq m \leq q\}$. We say that *consensus is reached asymptotically* if, for every $x_i(0) \in \mathbb{R}$, there exists $x^* \in [\min_{i\in\langle 1,N\rangle} x_i(0), \max_{i\in\langle 1,N\rangle} x_i(0)]$ such that, for all $i \in \langle 1, N\rangle$,

$$\lim_{k\to+\infty} x_i(k) = x^*. \tag{3}$$

The value of x^* depends on the agents' initial conditions but also on the evolution of the coefficients and on the sequence of delays. Naturally, consensus cannot be reached for arbitrary sequences of the coefficients or the delays.

In what follows, we introduce and discuss some standard assumptions that guarantee asymptotic consensus.

Assumption 1 (Nontrivial convex interaction)**.** There exists a positive constant $\alpha < 1$ (strength of interaction) such that, for all $i, j \in \langle 1, N\rangle$ and all $k \geq 0$, $A(k)$ satisfies:

1. $a_{ii}(k) \geq \alpha$;
2. $a_{ij}(k) \in \{0\} \cup [\alpha, 1)$;
3. $\sum_{j=1}^{N} a_{ij}(k) = 1$.

Item 1 of Assumption 1 states that $x_i(k)$ must be used in every iteration, while state variables from other agents may not be used as their availability is not ensured at every time step (item 2). Item 3 and the fact that all coefficients are nonnegative implies that the combination of state variables is always a convex one. This implies that at every time step $x_i(k) \in [\min_{i\in\langle 1,N\rangle} x_i(0), \max_{i\in\langle 1,N\rangle} x_i(0)]$.

Assumption 2 (Bounded discrete-time delays)**.** There exists a positive integer $\overline{\tau}$ such that for all $i, j \in \langle 1, N\rangle$ and all $k \geq 0$:

1. $\tau_{ii}(k) = 0$;

2. $0 \leq \tau_{ij}(k) \leq \overline{\tau}$.
3. $0 \leq k - \tau_{ij}(k)$;

Assumption 2 essentially states that each agent has always access to its current state (item 1) and that the time delays are non-negative and upper bounded by some constant (item 2). Item 3 is introduced so that the initial condition is given by $x(0)$. For convenience, if $a_{ij}(k) = 0$ then $\tau_{ij}(k) = k$.

The communication topology at each iteration can be described in terms of a directed graph $\mathcal{A}(k) = (\langle 1, N\rangle, E(k))$, where $(j, i) \in E(k)$ if and only if $a_{ij}(k) > 0$. That is, the structure of the directed graph $\mathcal{A}(k)$ and that of matrix $A(k)$ are linked. We thus say that graph $\mathcal{A}(k)$ is induced by $A(k)$.

Assumption 3 (Periodically rooted digraph)**.** For any sequence of directed graphs $\{\mathcal{A}(k)\}_{k=0}^{+\infty}$, there exists a positive constant B such that the union graph over any interval of length B is a rooted graph, that is, for all $k_0 \geq 0$,

$$\bigcup_{k=k_0}^{k_0+B-1} \mathcal{A}(k) \tag{4}$$

has at least one root.

Although the sequence of graphs may not be point-wise rooted, Assumption 3 guarantees that its union over a bounded time interval is a rooted graph. Given the previous setup, we have the following result.

Theorem 1. *Under Assumptions 1, 2, and 3, the discrete-time iterations described by (2) reach consensus asymptotically.*

This theorem can be proved using several techniques including those in, e.g., [7, 12, 14].

3. Continuous-time consensus with discrete-time updates

In this section, we start by introducing the problem of continuous-time consensus with discrete-time updates. We then present our proposed solution and introduce our main assumptions. Finally, we analyze the convergence properties of our solution using the results of the previous section.

Consider again N agents each one with its own state variable $x_i \in \mathbb{R}$ as before, except that now, instead of being discrete in time, the state variables must be continuous in time. Each agent has its own initial condition $x_i(t_0^i)$ and satisfies, for all $t \geq t_0^i$,

$$\dot{x}_i = u_i \tag{5}$$

where u_i is a control input to be specified. To avoid unnecessary complexity, we assume that $t_0^i = t_0$ for all $i \in \langle 1, N \rangle$. We say that *consensus is reached asymptotically* if, for every $x_i(t_0) \in \mathbb{R}$, there exists $x^* \in [\min_{i\in\langle 1,N\rangle} x_i(t_0), \max_{i\in\langle 1,N\rangle} x_i(t_0)]$ such that for all $i \in \langle 1, N \rangle$

$$\lim_{t\to+\infty} x_i(t) = x^*. \tag{6}$$

Naturally, to reach consensus agents must exchange information among them. Since continuous communication links among agents are hard to achieve in practice, we consider the case where the information exchange among agents occurs only at update times. Each agent has its own set of update times that are represented by an increasing sequence of time instants $\mathcal{T}_i = \{t_k^i\}_{k=0}^{+\infty}$ with $i \in \langle 1, N \rangle$. At each update time t_k^i, agent i receives information about its neighboring agents, a subset of all the agents that, in general, changes over time. Suppose that at time t_k agent i has access to the state of agent j, that is, $x_j(t_k^i)$ is available to agent i to use as it sees fit. The neighborhood of agent i is thus defined as

$$\mathcal{N}_i(t_k^i) = \{j \in \langle 1, N \rangle : x_j(t_k^i) \text{ is available to agent } i \text{ at time } t = t_k^i\}. \tag{7}$$

Furthermore, the information available to agent i may be received with latency due to a number of factors that include measurement and computation times and transmission delays. This latency is modeled a time delay. Let agent j be a neighbor of agent i. Due to latency, at time $t = t_k^i$, instead of having access to the current state of the neighboring agent $x_j(t_k^i)$, the information available to agent i is $x_j(t_k^i - \gamma_{ij}(t_k^i))$ where $\gamma_{ij}(t_k^i) \geq 0$ is a time delay. At this stage, agent i is allowed to perform some computations that must lead to consensus. Note that we are working in an asynchronous setup as each agent has is own set of update times and there are no synchronized clocks among agents. In this setup, each agent performs its own computations independently of the other agents.

The consensus problem addressed in this paper is formally stated as follows.

Problem 1. Given N identical agents described by the model (5), each with its own sequence of update times $\mathcal{T}_i$, find a distributed control strategy such that consensus is reached asymptotically under time-varying neighborhoods and time-varying delays.

In the above, by distributed control strategies we mean strategies that only require access the agent's own state information and to information received from neighboring agents. The conditions under which we derive a distributed control law yielding consensus are given later.

3.1. Proposed solution

We begin by introducing an additional state variable $X_i \in \mathbb{R}$, one for each agent, that may have discontinuities at update times, unlike x_i that must be continuous for all time. The additional variable X_i represents a value that x_i should track.

Between two update times of agent i, say t_k^i and t_{k+1}^i, the only information it has available is its own, that is, agent i only has access to the values of x_i and X_i. During this time interval, the evolution of x_i and X_i is dictated by

$$\dot{x}_i(t) = -b_i(t_k^i)\big(x_i(t) - X_i(t)\big) \tag{8}$$

$$\dot{X}_i(t) = c_i(t_k^i)\big(x_i(t) - X_i(t)\big) \tag{9}$$

where $b_i(t_k^i) > 0$ and $c_i(t_k^i) \geq 0$. This type of dynamics leads to a decrease of the absolute difference between x_i and X_i.

At each update time t_k^i, with $k \geq 1$, agent i receives the state variables of its neighboring agents. The value of X_i is updated using the information received that may be outdated, while x_i remains unchanged. Formally, we have the update equations

$$x_i(t_k^i) = x_i(t_k^{i-}) \tag{10}$$

$$X_i(t_k^i) = d_{ii}(t_k^i)X_i(t_k^{i-}) + \sum_{j\in\mathcal{N}_i(t_k^i)} d_{ij}(t_k^i)x_j(t_k^i - \gamma_{ij}(t_k^i)) \tag{11}$$

where $x(t^-) = \lim_{s\to 0^-} x(t+s)$. We use the convention that if $\gamma_{ij}(t_k^i) = 0$, then $t_k^i - \gamma_{ij}(t_k^i) = t_k^{i-}$.

Remark 1. The strategy proposed in [23] can be written for agent i as

$$\dot{x}_i(t) = \begin{cases} 0, & \mathcal{N}_i(t_k^i) = \emptyset \\ \sum_{j\in\mathcal{N}_i(t_k^i)} a_{ij}(t_k^i)\left(x_j(t_k^i - \gamma_{ij}(t_k^i)) - x_i(t)\right), & \text{otherwise} \end{cases}. \tag{12}$$

This type of dynamics may be written in our setup by making $b_i(t_k^i) = 1$ and $c_i(t_k^i) = 0$ in (8)-(9), and $d_{ij}(t_k^i) = a_{ij}(t_k^i)$ in (11) and $d_{ii}(t_k^i) = 0$. If $\mathcal{N}_i(t_k^i) = \emptyset$, then $b_i(t_k^i) = 0$. For our proof strategy to work, the weights b_i and d_{ii} must be positive (if this does not hold then Assumption 1 will not be satisfied; see the proof of Theorem 2 for more details). Although, we cannot claim that the results in [23] are a particular case of ours, note the existence of extra degrees of freedom in the form of weights b_i and c_i in (8)-(9) that are not present in (12).

3.2. Main assumptions

For the proposed strategy to work, some assumptions are required. These are introduced in the sequel. Given each sequence of individual update times $\mathcal{T}_i$, our strategy for a consensus proof requires us to construct a "larger" sequence of update times $\mathcal{T}$ formed by merging every individual sequence $\mathcal{T}_i$, that is,

$$\mathcal{T} = \{t_k\}_{k=0}^{+\infty} = \bigcup_{i=1}^{N} \mathcal{T}_i. \tag{13}$$

Merging every sequence of update times requires repeated update times to be deleted and the sequence reordered so that the update times are in increasing order. We begin by considering the following property that we would like $\mathcal{T}$ to satisfy.

Property 1 (Bounded communication intervals)**.** Given an increasing sequence of time instants $\{s_k\}_{k=0}^{+\infty}$, there exist positive constants $\underline{\delta}$ and $\overline{\delta}$ such that $0 < \underline{\delta} \leq s_{k+1} - s_k \leq \overline{\delta} < +\infty$ holds for all $k \geq 0$.

It turns out that even if each $\mathcal{T}_i$ satisfies Property 1, $\mathcal{T}$ may not inherit this property as shown by the following counterexample.

Counterexample. Let $N = 2$. Given $\delta > 0$, let

$$\mathcal{T}_1 = \{t_0^1 = 0; t_1^1 = t_0^1 + 2\delta; t_{k+1}^1 = t_k^1 + \delta, \text{ for } k \geq 1\} \tag{14}$$

and

$$\mathcal{T}_2 = \{t_0^2 = 0; t_1^2 = t_0^2 + \frac{3}{2}\delta; t_{k+1}^2 = t_k^2 + \left(1 + \frac{1}{2^{k+1}}\right)\delta, \text{ for } k \geq 1\}. \tag{15}$$

Note that $\delta \leq t^1_{k+1} - t^1_k \leq 2\delta$ and $\delta \leq t^2_{k+1} - t^2_k \leq \frac{3}{2}\delta$ for all $k \geq 0$. Hence, both $\mathcal{T}_1$ and $\mathcal{T}_2$ satisfy Property 1. Let

$$\mathcal{T} = \mathcal{T}_1 \cup \mathcal{T}_2 = \{t_k\}_{k=0}^{+\infty} = \{0, t^2_1, t^1_1, t^2_2, t^1_2, \ldots\}. \tag{16}$$

Since

$$t^1_k - t^2_k = \frac{1}{2^{k+1}} \to 0, \tag{17}$$

there exists no $\underline{\delta} > 0$ such that $t_{k+1} - t_k \geq \underline{\delta}$ for all $k \geq 0$. Hence, $\mathcal{T}$ does not satisfy Property 1.

Since, in our proof of convergence, the existence of a lower bound $\underline{\delta}$ in Property 1 for the sequence $\mathcal{T}$ is crucial, we consider the following assumption.

Assumption 4 (Quantized communication intervals)**.** Given $\Delta > 0$ and $M \in \mathbb{N}$, each sequence of update times $\mathcal{T}_i$ with $i \in \langle 1, N \rangle$ is such that, for all $k \geq 0$,

$$t^i_{k+1} - t^i_k \in \{q\Delta : q \in \langle 1, M \rangle\}. \tag{18}$$

Assumption 4 states that the update intervals of each agent are quantized, that is, they must be integer multiples (up to a maximum of M) of a fundamental update interval Δ. If Assumption 4 is satisfied, then $\mathcal{T}$ satisfies Property 1[1]. Assumption 4 is not very restrictive as one may take M arbitrarily large to approximate any bounded interval with arbitrarily high accuracy.

Given $\mathcal{T}$, we now extend the coefficients in (8)-(9) and (10)-(11), that were only defined at update times of each individual agent, to be defined over the whole union sequence. For all $i \in \langle 1, N \rangle$ and all $t_k \in \mathcal{T}$, let

$$b_i(t_k) = b_i(t^i_p) \tag{19}$$

$$c_i(t_k) = c_i(t^i_p) \tag{20}$$

[1] This statement is a particular case of Lemma 2 to be introduced later in the text (Section 3.4).

where $p = \max\{r \geq 0 : t_r^i \leq t_k\}$. For all $i, j \in \langle 1, N\rangle$ and all $t_k \in \mathcal{T}$, let

$$d_{ii}(t_k) = \begin{cases} d_{ii}(t_p^i), & \text{if } t_k = t_p^i \in \mathcal{T}_i \\ 1, & \text{otherwise} \end{cases} \tag{21}$$

$$d_{ij}(t_k) = \begin{cases} d_{ij}(t_p^i), & \text{if } t_k = t_p^i \in \mathcal{T}_i \\ 0, & \text{otherwise} \end{cases}. \tag{22}$$

Assumption 5. There exist $\overline{b}, \underline{b}, \overline{c}, \underline{c} > 0$ such that, for all $i \in \langle 1, N\rangle$ and all $k \geq 0$, $b_i(t_k) \in [\underline{b}, \overline{b}]$ and $c_i(t_k) \in \{0\} \cup [\underline{c}, \overline{c}]$.

Let $D_k = D(t_k)$ denote the matrix whose entries are $d_{ij}(t_k)$. We assume that the sequence $\{D_k\}_{k=0}^{+\infty}$ satisfies Assumption 1 with strength of interaction $\beta \in (0, 1)$.

To each time instant, t_k, we associate a directed graph $\mathcal{G}_k = \mathcal{G}(t_k) = (V, E_k)$ with $V = \langle 1, N\rangle$ and $E_k \subseteq V \times V$ defined as $E_k = \{(j, i) : d_{ij}(t_k) > 0, i, j \in \langle 1, N\rangle\}$. Each graph $\mathcal{G}_k$ is induced by the matrix D_k and captures the information available to each agent at time t_k. We assume that the sequence of graphs $\{\mathcal{G}_k\}_{k=0}^{+\infty}$ satisfies Assumption 3.

As previously mentioned, the information received by each agent may be outdated due to a number of factors. The latencies are modeled as time delays γ_{ij} introduced in (11). Similar to what was done for the coefficients d_{ij}, we extend the time delays so that they are defined for all $t_k \in \mathcal{T}$, by letting

$$\gamma_{ij}(t_k) = \begin{cases} \gamma_{ij}(t_p^i), & \text{if } t_k = t_p^i \in \mathcal{T}_i \\ 0, & \text{otherwise} \end{cases} \tag{23}$$

for all $i, j \in \langle 1, N\rangle$. The time delays are assumed to satisfy the following:

Assumption 6 (Quantized time delays)**.** Given $\Delta_d > 0$ and $M_d \in \mathbb{N}$, the sequence of delays is such that, for all $k \geq 0$ and all $1 \leq i, j \leq N$:

1. $\gamma_{ii}(t_k^i) = 0$;
2. $\gamma_{ij}(t_k^i) \in \{q\Delta_d : q \in \langle 0, M_d\rangle\}$;
3. $t_0 \leq t_k^i - \gamma_{ij}(t_k^i)$.

The reason for considering quantized delays (item 2) stems, as previously mentioned, from the need to guarantee the existence of a lower bound $\underline{\delta}$ in Property 1 for the sequence $\mathcal{T}$.

3.3. State augmentation

Let the state of the whole system formed by the N agents be represented by

$$z(t) = \begin{bmatrix} x_1(t) & X_1(t) & \cdots & x_N(t) & X_N(t) \end{bmatrix}^\top, \tag{24}$$

with initial state $z(t_0) \in \mathbb{R}^{2N}$. The dynamics of either variable (x_i or X_i) may depend on the values of other state variables. This dependence is represented by a graph, an interaction graph, with vertex set $\overline{V} = \langle 1, 2N \rangle$ (one vertex for each variable). Odd vertices are associated to x_i variables while even vertices are associated to X_i variables.

The continuous dynamics (8)-(9) can be written in terms of the augmented state variable z as

$$\dot{z}(t) = L(t_k) z(t) \tag{25}$$

for all $t \in [t_k,\, t_{k+1})$, where

$$L(t_k) = \operatorname{diag}(L_1(t_k), L_2(t_k), \ldots, L_N(t_k)) \tag{26}$$

with

$$L_i(t_k) = \begin{bmatrix} -b_i(t_k) & b_i(t_k) \\ c_i(t_k) & -c_i(t_k) \end{bmatrix}. \tag{27}$$

For all $\delta \geq 0$, $b > 0$, and $c \geq 0$, let

$$\Psi(\delta, b, c) = \frac{1}{b+c} \begin{bmatrix} c + b f(\delta, b, c) & b(1 - f(\delta, b, c)) \\ c(1 - f(\delta, b, c)) & b + c f(\delta, b, c) \end{bmatrix} \tag{28}$$

where $f(\delta, b, c) = e^{-(b+c)\delta}$. For all $k \geq 0$, (25) implies that

$$z(t_{k+1}^-) = \Phi(t_{k+1}, t_k) z(t_k), \tag{29}$$

where, for all $t \geq s$,

$$\Phi(t, s) = \operatorname{diag}(\Phi_1(t, s), \Phi_2(t, s), \ldots, \Phi_N(t, s)) \tag{30}$$

with

$$\Phi_i(t, s) = \exp\{L_i(t_k)(t - s)\} = \Psi(t - s, b_i(t_k), c_i(t_k)). \tag{31}$$

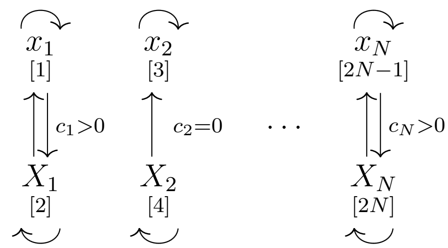


Figure 1: Interaction graph, denoted by $\mathcal{H}_k$, among state variables for $t \in [t_k,\, t_{k+1})$ and induced by matrix $\Phi(t_{k+1}, t_k)$. Numbers in brackets represent the vertex in $\overline{V}$ associated to each state variable. Edges departing from x_i and entering X_i are only present if $c_i(t_k) > 0$.

Let $\mathcal{F} = (\overline{V}, \overline{F})$ where

$$\overline{F} = \{(2i, 2i-1) : i \in V\} \cup \{(i,i) : i \in \overline{V}\}. \tag{32}$$

The graph induced by $\Phi(t_{k+1}, t_k)$ is denoted by $\mathcal{H}_k = (\overline{V}, \overline{H}_k)$, where

$$\overline{H}_k = \overline{F} \cup \{(2i-1, 2i) : i \in V \wedge c_i(t_k) > 0\}, \tag{33}$$

as depicted in Figure 1.

At each update time t_k, with $k \geq 1$, using the aggregated state, we can write the update equations (10)-(11) as

$$z_i(t_k) = \sum_{j=1}^{2N} r_{ij}(t_k) z_j(t_k - \sigma_{ij}(t_k)), \tag{34}$$

where, for all $i, j \in \overline{V}$,

$$r_{ij}(t_k) = \begin{cases} 1, & \text{if } i = j = 2p-1 \\ d_{pp}(t_k), & \text{if } i = j = 2p \\ d_{pq}(t_k), & \text{if } i = 2p \text{ and } j = 2q-1 \\ 0, & \text{otherwise} \end{cases} \tag{35}$$

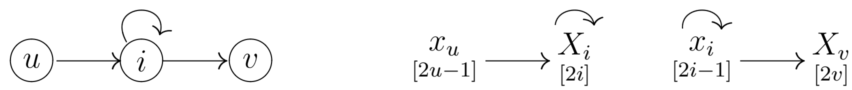
and

$$\sigma_{ij}(t_k) = \begin{cases} \gamma_{pq}(t_k), & \text{if } i = 2p \text{ and } j = 2q-1 \\ 0, & \text{otherwise} \end{cases}. \tag{36}$$

Let $R_k = R(t_k)$ denote the matrix whose entries are $r_{ij}(t_k)$. The graph induced by R_k is denoted by $\overline{\mathcal{G}}_k = (\overline{V}, \overline{E}_k)$, where

$$\overline{E}_k = \{(2i-1, 2j) : (i,j) \in E_k \wedge i \neq j\} \cup \{(i,i) : i \in \overline{V}\}, \tag{37}$$

with $\mathcal{G}_k = (V, E_k)$ denoting the graph induced by the matrix D_k. See Figure 2 for a graphical illustration of the relation between $\mathcal{G}_k$ and $\overline{\mathcal{G}}_k$.



(a) Information flow among agents u, i, and v according to $\mathcal{G}_k$.

(b) Interaction graph among state variables in $\overline{\mathcal{G}}_k$.

Figure 2: Relation between $\mathcal{G}_k$ and $\overline{\mathcal{G}}_k$. Agent i has one incoming arc from u and one outgoing arc to v in $\mathcal{G}_k$, that give rise to the arcs $(2u-1, 2i)$ and $(2i-1, 2v)$, respectively, in $\overline{\mathcal{G}}_k$. Only self-arcs associated with i are represented.

3.4. Main result

The following result establishes that the strategy described in Section 3.1 under the assumptions presented in Section 3.2 solves Problem 1.

Theorem 2. *Consider the dynamical system with state z driven by equations (25) and (34). If*

1. *Assumptions 4, 5, and 6 are satisfied;*
2. *the sequence of matrices $\{D_k\}_{k=0}^{+\infty}$ satisfies Assumption 1; and,*
3. *the sequence of directed graphs $\{\mathcal{G}_k\}_{k=0}^{+\infty}$ satisfies Assumption 3;*

then z reaches consensus asymptotically.

The proof of the theorem consists of constructing an equivalent discrete-time description of the system, applying Theorem 1 to this system, and concluding that consensus is reached asymptotically. To illustrate the proof strategy, consider the case without time delays. Let $y(p) \in \mathbb{R}^{2N}$ be a new discrete-time state variable introduced for analysis purposes only and defined as $y(p) = z(t_{p/2})$ for $p = 0, 2, 4, \ldots$, and $y(p) = z(t^-_{(p+1)/2})$ for $p = 1, 3, 5, \ldots$. We can further write

$$y(p+1) = A(p)y(p), \tag{38}$$

where the matrix $A(p)$ is defined as

$$A(p) = \begin{cases} \Phi(t_{\frac{p}{2}+1}, t_{\frac{p}{2}}), & \text{if } p = 0, 2, 4, \ldots \\ R_{\frac{p+1}{2}}, & \text{if } p = 1, 3, 5, \ldots \end{cases}. \tag{39}$$

The diagram in Figure 3 depicts the sequence of iterations that variable y undergoes. If $\mathcal{T}$ satisfies Property 1 then, for all $k \geq 0$, the entries of $\Phi(t_{k+1}, t_k)$ satisfy Assumption 1 with strength of interaction ϕ for some $\phi \in$

$$\begin{array}{ccccccc} y(0)=z(t_0) & & y(2)=z(t_1) & & y(4)=z(t_2) & & \\ \big\downarrow{\scriptstyle \Phi(t_1,t_0)} & \nearrow_{R_1} & \big\downarrow{\scriptstyle \Phi(t_2,t_1)} & \nearrow_{R_2} & \big\downarrow{\scriptstyle \Phi(t_3,t_2)} & \nearrow_{R_3} & \cdots \\ y(1)=z(t_1^-) & & y(3)=z(t_2^-) & & y(5)=z(t_3^-) & & \end{array}$$

Figure 3: Sequence of iterations performed by the variable y.

$(0,1)$. Note that R_k satisfies Assumption 1 with strength of interaction $\beta \in (0,1)$ because the same holds for D_k. Therefore, the entries of $A(p)$ satisfy Assumption 1 with strength of interaction $\alpha = \min\{\phi, \beta\}$. Let $\mathcal{A}(p)$ denote the graphs induced by the matrix $A(p)$. For all $p_0 \geq 0$, we have that

$$\bigcup_{p=p_0}^{p_0+2B-1} \mathcal{A}(p) = \bigcup_{q=q_0}^{q_0+B-1} \mathcal{H}_q \cup \bigcup_{k=k_0}^{k_0+B-1} \overline{\mathcal{G}}_k \tag{40}$$

where

$$(q_0, k_0) = \begin{cases} \left(\frac{p_0}{2}, \frac{p_0}{2}+1\right), & \text{if } p_0 = 0, 2, 4, \ldots \\ \left(\frac{p_0+1}{2}, \frac{p_0+1}{2}\right), & \text{if } p_0 = 1, 3, 5, \ldots \end{cases}. \tag{41}$$

If (40) is a rooted graph, then the sequence of graphs $\{\mathcal{A}(p)\}_{p=0}^{+\infty}$ is periodically rooted with period $2B$. The following lemma shows that, if the sequence of directed graphs $\{\mathcal{G}_k\}_{k=0}^{+\infty}$ is periodically rooted with period B, then (40) is indeed a rooted graph.

Lemma 1. *Given $k_0, q_0 \geq 0$ and $B \geq 1$, if $\bigcup_{k=k_0}^{k_0+B-1} \mathcal{G}_k$ is rooted, then $\bigcup_{q=q_0}^{q_0+B-1} \mathcal{H}_q \cup \bigcup_{k=k_0}^{k_0+B-1} \overline{\mathcal{G}}_k$ is also rooted.*

Proof. Let $r \in V$ be a root of $\mathcal{J}_1 = \bigcup_{k=k_0}^{k_0+B-1} \mathcal{G}_k$. We will show that $2r \in \overline{V}$ is a root of $\mathcal{J}_2 = \bigcup_{q=q_0}^{q_0+B-1} \mathcal{H}_q \cup \bigcup_{k=k_0}^{k_0+B-1} \overline{\mathcal{G}}_k$, which is equivalent to showing that there exists a path in $\mathcal{J}_2$ from $2r$ to every other vertex in $\overline{V}$. Let $\overline{v} \in \overline{V}$ and

$$V \ni v = \begin{cases} \frac{\overline{v}}{2}, & \text{if } \overline{v} \text{ is even} \\ \frac{\overline{v}+1}{2}, & \text{if } \overline{v} \text{ is odd} \end{cases}. \tag{42}$$

Since r is a root of $\mathcal{J}_1$, there exists a path in $\mathcal{J}_1$ from r to v for all $v \in V$. Let $\{r, u_1, u_2, \ldots, u_l, v\}$ be one such path. Note that $\mathcal{F} \subseteq \bigcup_{q=q_0}^{q_0+B-1} \mathcal{H}_q$. We will now construct a path in $\mathcal{J}_2$ from $2r$ to $\overline{v}$ for any $\overline{v} \in \overline{V}$, using the facts that, for all $i, j \in \langle 1, N \rangle$, all arcs of the form $(2u_i, 2u_i - 1)$ belong to $\mathcal{F}$, and that arcs (u_i, u_j) in $\mathcal{G}_k$ map to arcs $(2u_i - 1, 2u_j)$ in $\overline{\mathcal{G}}_k$. If $\overline{v}$ is even, then

$$\{2r, 2r-1, 2u_1, 2u_1-1, 2u_2, 2u_2-1, \ldots, 2u_l, 2u_l-1, 2v\} \tag{43}$$

is a path in $\mathcal{J}_2$ from $2r$ to $\overline{v}$. If $\overline{v}$ is odd, then the path becomes (43) with $2v-1$ added at the end. $\square$

Thus, in the absence of time delays, Theorem 1 guarantees that y reaches consensus asymptotically.

To accommodate for time delays in continuous-time, terms of the form $z_j(t_k - \sigma_{ij}(t_k))$ must correspond to some $y_j(p - \tau_{ij}(p))$, that is, each continuous delay $\sigma_{ij}(t_k)$ needs to be translated into an integer delay $\tau_{ij}(p)$. In order to accomplish this, each instance of $t_k - \sigma_{ij}(t_k)$ is added to the existing sequence of update times (removing duplicates and reordering if necessary), thus generating a new time instant that we shall refer to as a *delay event*. By reordering the resulting sequence, we get an extended sequence of time instants $\widehat{\mathcal{T}} = \{\hat{t}_k\}_{k=0}^{+\infty}$ of the form

$$t_0 = \hat{t}_{m_0=0} < \hat{t}_1 < \hat{t}_2 < \cdots < \hat{t}_{m_1-1} < \hat{t}_{m_1} = t_1 < \hat{t}_{m_1+1} < \cdots \tag{44}$$

where $\{m_k\}_{k=0}^{+\infty}$ is a sequence of indices such that $\hat{t}_{m_k} = t_k$ for all $k \geq 0$. The delay events are the time instants $\hat{t}_q$ with $q \geq 0$ such that $q \neq m_k$ for all $k \geq 0$. At each delay event, an update is performed that preserves every value, that is, for all $0 \leq q \neq m_k$, we have

$$z(\hat{t}_q) = z(\hat{t}_q^-). \tag{45}$$

Next, we show that the extended sequence $\widehat{\mathcal{T}}$ satisfies Property 1.

Lemma 2. *If Assumptions 4 and 6 are satisfied, then $\widehat{\mathcal{T}} = \bigcup_{i=1}^{N} \widehat{\mathcal{T}}_i$ satisfies Property 1.*

Proof. First, note that $\{\sigma_{ij}(t_k)\}$ satisfies Assumption 6 because $\{\gamma_{ij}(t_k)\}$ also satisfies that same assumption. Clearly, for all $k \geq 0$ and all $i \in \langle 1, N \rangle$, we have $\hat{t}_{k+1} - \hat{t}_k \leq M\Delta$. Thus, let $\overline{\delta} = M\Delta$. To prove that a lower bound for $\hat{t}_{k+1} - \hat{t}_k$ exists, we begin by observing that every element in $\widehat{\mathcal{T}}$ can be written as

$$t_0 + \Delta u - \Delta_d v \tag{46}$$

where $u \in \mathbb{Z}_0^+$ and $v \in \langle 0, M_d \rangle$. Let $\hat{t}_k, \hat{t}_p \in \widehat{\mathcal{T}}$ and assume, without loss of generality, that $\hat{t}_k > \hat{t}_p$. The difference $\hat{t}_k - \hat{t}_p$ is always equal to or greater than

$$\begin{aligned}
\underline{\delta} &= \min_{\substack{u_1,u_2 \in \mathbb{Z}_0^+ \\ v_1,v_2 \in \langle 0,M_d \rangle \\ \Delta u_1 - \Delta_d v_1 > \Delta u_2 - \Delta_d v_2}} t_0 + \Delta u_1 - \Delta_d v_1 - (t_0 + \Delta u_2 - \Delta_d v_2) && (47) \\
&= \min_{\substack{e \in \mathbb{Z} \\ f \in \langle 0,M_d \rangle \\ \Delta e + \Delta_d f > 0}} \Delta e + \Delta_d f. && (48)
\end{aligned}$$

Solving (48) for e, yields

$$\underline{\delta} = \min_{\substack{f \in \langle 0,M_d \rangle \\ \mathrm{rem}(\Delta_d f, \Delta) > 0}} \min\{\mathrm{rem}(\Delta_d f, \Delta), \Delta - \mathrm{rem}(\Delta_d f, \Delta)\} \tag{49}$$

where $\mathrm{rem}(\cdot,\cdot)$ is the remainder after division[2]. Note that $\underline{\delta}$ in (49) always exists since the minimization is over a finite set. We conclude that $\widehat{\mathcal{T}}$ satisfies Property 1. $\square$

A natural question that arises is how many delay events are added to each interval of the form (t_k, t_{k+1}). Let n_k denote the number of delay events that belong to the time interval (t_k, t_{k+1}). Assumption 6 implies the time delays are bounded, and therefore only a finite number of delay events take place on any time interval (t_k, t_{k+1}), as shown by the following lemma.

Lemma 3. *The number of delay events on any interval (t_k, t_{k+1}) is upper bounded by*

$$\overline{n} = \left(\left\lfloor \frac{M_d \Delta_d}{\underline{\delta}} \right\rfloor + 1 \right) N(N-1), \tag{50}$$

where $\lfloor x \rfloor$ stands for the largest integer less than or equal to x.

Proof. Fix some $k \geq 0$. Only update times equal to or larger than t_{k+1} can generate delay events that fall in the time interval (t_k, t_{k+1}). Since, by Assumption 6, the delays are upper bounded by $M_d \Delta_d$, there will be an update time t_p satisfying $t_p \leq t_{k+1} + M_d \Delta_d < t_{p+1}$ such that t_{p+1} cannot

[2]For $x \in \mathbb{R}$ and $y \in \mathbb{R}^+$, $\mathrm{rem}(x, y) = x - py$ where $p = \lfloor x/y \rfloor$.

generate delay events belonging to (t_k, t_{k+1}). The number of update times in the interval $[t_{k+1}, t_p]$ is $p-k$ that is bounded by

$$p - k \leq \left\lfloor \frac{t_p - t_{k+1}}{\underline{\delta}} \right\rfloor + 1 \leq \left\lfloor \frac{M_d \Delta_d}{\underline{\delta}} \right\rfloor + 1. \tag{51}$$

where we used the fact that $t_p - t_{k+1} \leq M_d \Delta_d$. The generation of delay events is maximized when the time instants $t_l - \gamma_{ij}(t_l)$ are different for all $i, j \in \langle 1, N \rangle$ with $i \neq j$ and for all $l \in \langle k+1, p \rangle$. In this case, each update time generates $N(N-1)$ distinct delay events that fall in the time interval (t_k, t_{k+1}). We conclude that the maximum number of delay events in (t_k, t_{k+1}) is less than or equal to $\overline{n}$ defined in (50). Noting that (50) is independent of k, the proof is complete. □

We are now ready to prove our main result.

Proof of Theorem 2. In order to apply the discrete-time consensus result, we introduce a new state variable $y \in \mathbb{R}^{2N}$, defined as

$$y(p) = \begin{cases} z\big(\hat{t}_{p/2}\big), & \text{if } p = 0, 2, 4, \ldots \\ z\big(\hat{t}^-_{(p+1)/2}\big), & \text{if } p = 1, 3, 5, \ldots \end{cases}. \tag{52}$$

In the sequel, we will show that this state variable evolves in discrete-time, according to

$$y_i(p+1) = \sum_{j=1}^{2N} a_{ij}(p) y_j(p - \tau_{ij}(p)), \tag{53}$$

where a_{ij} and τ_{ij} are defined in the sequel. The main idea of the proof is to show that y, driven by (53), satisfies all conditions necessary for asymptotical consensus as required by Theorem 1, and that this implies that z also reaches consensus asymptotically.

Let $A(p)$ denote the matrix whose entries are $a_{ij}(p)$ and let $\mathcal{A}(p)$ denote the graph induced by each matrix. Depending on p, matrix $A(p)$ can be R_k, $\Phi(\hat{t}_{k+1}, \hat{t}_k)$, or I_{2N}, with induced graphs $\overline{\mathcal{G}}_k$, $\mathcal{H}_k$, or $\mathcal{G}^0$, respectively. Here, $\mathcal{G}^0 = (\overline{V}, \{(i,i) : i \in \overline{V}\})$ is a graph with all (and only) self-arcs (induced by I_{2N}, the identity matrix of dimension $2N$). Formally, $A(p)$ is defined as

$$A(p) = \begin{cases} \Phi\big(\hat{t}_{\frac{p}{2}+1}, \hat{t}_{\frac{p}{2}}\big), & \text{if } p = 0, 2, 4, \ldots \\ R_k, & \text{if } p = 1, 3, 5, \ldots \wedge \frac{p+1}{2} = m_k \\ I_{2N}, & \text{otherwise} \end{cases}. \tag{54}$$

Since $\widehat{\mathcal{T}}$ satisfies Property 1, the entries of $\Phi(t_{k+1}, t_k)$ satisfy Assumption 1 with strength of interaction

$$\phi = \min \Psi_{ij}(\delta, b, c) \tag{55}$$

$$\text{s. t. } i, j \in \langle 1, 2 \rangle, \delta \in [\underline{\delta}, \overline{\delta}], b \in [\underline{b}, \overline{b}], c \in [\underline{c}, \overline{c}] \tag{56}$$

where Ψ is defined in (28). Since (55) is an optimization over a compact set and the four objective functions are positive over that same set, ϕ always exists and is positive[3]. The matrix R_k satisfies Assumption 1 with strength of interaction $\beta \in (0, 1)$ because D_k also satisfies that same assumption. Finally, the identity matrix satisfies Assumption 1 for any strength of interaction in $(0, 1)$. Therefore, for all $p \geq 0$, the entries of $A(p)$ satisfy Assumption 1 with strength of interaction $\alpha = \min\{\phi, \beta\}$.

The discrete delays τ_{ij} in (53) are defined as follows (see also Figure 4). At each update time t_k, with $k \geq 1$, in order to compute $z_i(t_k)$ we need, among others, the value of $z_j(t_k - \sigma_{ij}(t_k))$. According to (52), this corresponds to the computation of $y_i(2m_k)$ because $t_k = \hat{t}_{m_k}$. If $\sigma_{ij}(t_k) = 0$, then the value of $z_j(t_k - \sigma_{ij}(t_k)) = z_j(t_k^-)$ is stored in $y_j(2m_k - 1)$ and thus the discrete delay is $\tau_{ij}(2m_k - 1) = 0$. If $\sigma_{ij}(t_k) > 0$, let $q_{ij} \geq 0$ be such that $\hat{t}_{q_{ij}} = t_k - \sigma_{ij}(t_k)$. Then, $z_j(t_k - \sigma_{ij}(t_k)) = z_j(\hat{t}_{q_{ij}})$. In terms of the discrete-time variable y, we are trying to access $y_j(2q_{ij})$ and hence $\tau_{ij}(2m_k - 1) = 2m_k - 1 - 2q_{ij} = 2(m_k - q_{ij}) - 1$. Formally, the delays $\tau_{ij}(p)$ are defined, for all $i, j \in \overline{V}$, as

$$\tau_{ij}(p) = \begin{cases} 2(m_k - q_{ij}(p)) - 1, & \text{if } p = 2m_k - 1 \text{ and } \sigma_{ij}(t_k) > 0 \\ 0, & \text{otherwise} \end{cases} \tag{57}$$

where $q_{ij}(p) \geq 0$ is such that $t_k - \sigma_{ij}(t_k) = \hat{t}_{q_{ij}(p)}$.

Next, we show that $\{\tau_{ij}(p)\}_{i,j=1}^N$ satisfies Assumption 2. For $p = 2m_k - 1$, it is easy to see that $\tau_{ii}(p) = 0$, $\tau_{ij}(p) \geq 1 \geq 0$ and that $p \geq p - 1 \geq p - \tau_{ij}(p) \geq 0$. All that is left to show is that the delays are bounded. Suppose $\sigma_{ij}(t_k) > 0$ and let $r \geq 0$ be such that

$$t_r = \hat{t}_{m_r} \leq t_k - \sigma_{ij}(t_k) = \hat{t}_{q_{ij}} < t_{r+1} = \hat{t}_{m_{r+1}} \leq t_k = \hat{t}_{m_k}. \tag{58}$$

Note that the sequence of indices $\{m_k\}_{k=0}^{+\infty}$ can be obtained through the recursion $m_{k+1} = m_k + n_k + 1$ starting with $m_0 = 0$ where n_k is the number

[3]If $\underline{\delta}$ was allowed to be zero, then $\phi = 0$ and $A(p)$ would not satisfy Assumption 1, thus preventing us from resorting to Theorem 1 to prove asymptotic consensus.

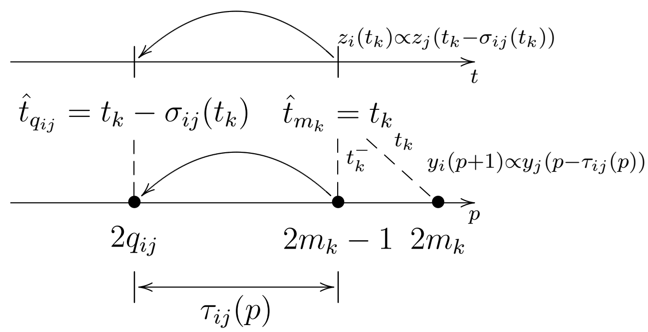


Figure 4: Definition of $\tau_{ij}(p)$ when $p = 2m_k - 1$ and $\sigma_{ij}(t_k) > 0$.

of delay events in (t_k, t_{k+1}). From this recursion, we have that

$$m_k - q_{ij} \leq m_k - m_r = \sum_{l=r}^{k-1} n_l + k - r \leq (\overline{n} + 1)(k - r) \tag{59}$$

where $\overline{n}$ is defined in (50). The quantity $k - r$ is equal to the number of update times in $[t_{r+1}, t_k]$ and is bounded by

$$k - r \leq \left\lfloor \frac{t_k - t_{r+1}}{\underline{\delta}} \right\rfloor + 1 \leq \left\lfloor \frac{M_d \Delta_d}{\underline{\delta}} \right\rfloor + 1, \tag{60}$$

where we have used the fact that $t_k - t_{r+1} < \sigma_{ij}(t_k) \leq M_d \Delta_d$. Replacing (60) in (59), we obtain

$$m_k - q_{ij} \leq (\overline{n} + 1) \left(\left\lfloor \frac{M_d \Delta_d}{\underline{\delta}} \right\rfloor + 1 \right). \tag{61}$$

Hence, we have that

$$\tau_{ij}(p) = 2(m_k - q_{ij}(p)) - 1 \leq 2(\overline{n} + 1) \left(\left\lfloor \frac{M_d \Delta_d}{\underline{\delta}} \right\rfloor + 1 \right) - 1 = \overline{\tau}. \tag{62}$$

In what follows, we show that the sequence of graphs $\{\mathcal{A}(p)\}_{p=0}^{+\infty}$ is periodically rooted with period $\overline{B} = 2(\overline{n} + 1)B$. Any sequence of such graphs with length $2(\overline{n} + 1)$ contains at least one $\overline{\mathcal{G}}_k$ graph. Thus, any sequence of length $\overline{B}$ contains at least of B graphs $\overline{\mathcal{G}}_k$. Using the fact that $\mathcal{G} \cup \mathcal{G} = \mathcal{G}$ for any graph $\mathcal{G}$, and that $\mathcal{G}^0 \cup \mathcal{H}_k = \mathcal{H}_k$, the union graph across any sequence of length $\overline{B}$ is equal to

$$\bigcup_{p=p_0}^{p_0+\overline{B}-1} \mathcal{A}(p) = \bigcup_{q=q_0}^{q_0+B'-1} \mathcal{H}_q \cup \bigcup_{k=k_0}^{k_0+B'-1} \overline{\mathcal{G}}_k \tag{63}$$

where $p_0 \geq 0$, $B' \geq B$, and (q_0, k_0) is given by (41). Lemma 1 guarantees that (63) has at least one root. We conclude that the sequence of graphs is periodically rooted with period $\overline{B}$, thus satisfying Assumption 3.

We conclude by Theorem 1 that y reaches consensus asymptotically. Therefore, there exists $y^* \in \mathbb{R}$ such that

$$\lim_{p\to+\infty} y(p) = y^* \mathbf{1}_{2N} \Rightarrow \lim_{k\to+\infty} z(t_k) = y^* \mathbf{1}_{2N} \tag{64}$$

where $\mathbf{1}_{2N} \in \mathbb{R}^{2N}$ is a vector with all entries equal to one. Notice that $z(t) = \Phi(t, t_k) z(t_k)$ for $t \in [t_k, t_{k+1})$. Since $\mathcal{T}$ satisfies Property 1, we have the following two facts: i) the sequence of update times diverges and therefore $t \to +\infty$ implies that $t_k \to +\infty$; and, ii) the entries of $\Phi(t, t_k)$ are bounded for all $k \geq 0$. Using these two observations and the fact that $\Phi(t, t_k)\mathbf{1}_{2N} = \mathbf{1}_{2N}$ for all $t \geq t_k$ and all $k \geq 0$, we conclude that

$$\lim_{t\to+\infty} z(t) = y^* \mathbf{1}_{2N}. \tag{65}$$

□

4. Illustrative example

In this section we provide an example that illustrates the proposed solution. Consider $N = 5$ agents whose initial states are $x_i(t_0) = X_i(t_0) = i - 3$ with $i \in V = \langle 1, 5 \rangle$. Each sequence of time instants is generated by taking $t_0^i = 0$ and randomly selecting (with a uniform distribution) the update intervals $t_{k+1}^i - t_k^i$ from (18) with $\Delta = 0.1$ and $M = 20$. The time delays are also generated randomly according to Assumption 6 with $\Delta_d = \frac{12}{34}\Delta \approx 0.0353$ and $M_d = 200$. Under this setup, we obtain $\underline{\delta} = \frac{1}{170} \approx 5.882 \times 10^{-3}$ and $\overline{\delta} = 2$ (according to Lemma 2). The communication among agents occurs as follows (see also Figure 5), for $k \geq 1$:

1. $\mathcal{N}_1(t_{4k}^1) = \{2\}$;
2. $\mathcal{N}_2(t_{4k+1}^2) = \{3\}$ and $\mathcal{N}_2(t_{4k+3}^2) = \{1\}$;

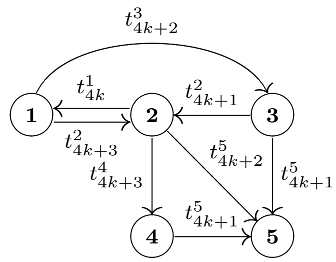


Figure 5: Union graph with update times associated to each edge. Agents 1, 2, and 3 are all roots of the graph.

3. $\mathcal{N}_3(t^3_{4k+2}) = \{1\}$;
4. $\mathcal{N}_4(t^4_{4k+3}) = \{2\}$;
5. $\mathcal{N}_5(t^5_{4k+1}) = \{3, 4\}$ and $\mathcal{N}_5(t^5_{4k+2}) = \{2\}$.

Where unspecified, the neighborhoods are defined as empty. The resulting sequence of graphs is periodically rooted with period $B \geq 4M = 80$ or, in terms of time, is rooted on every interval of length at least $4M\Delta = 8$. The values of the coefficients of the continuous dynamics are, for all $k \geq 0$, $b_i(t^i_k) = 1$ and $c_i(t^i_k) = 0$ for $i \in \langle 1, 2\rangle$ and $b_i(t^i_k) = \frac{4}{5}$ and $c_i(t^i_k) = \frac{1}{5}$ for $i \in \langle 3, 5\rangle$. In the discrete updates, for each $i \in V$, if $\mathcal{N}_i(t^i_k) = \emptyset$, then $d_{ii}(t^i_k) = 1$; otherwise, $d_{ii}(t_k)$ is drawn randomly from the interval $[\alpha, 1-|\mathcal{N}_i(t_k)|\alpha]$ where $|\mathcal{X}|$ denote the number of elements of a given set $\mathcal{X}$. The remaining coefficients are given by

$$d_{ij}(t_k) = \begin{cases} \frac{1-d_{ii}(t_k)}{|\mathcal{N}_i(t_k)|}, & \text{if } |\mathcal{N}_i(t_k)| > 0 \\ 0, & \text{otherwise} \end{cases} \tag{66}$$

for all $i, j \in V$ with $i \neq j$. The value of α is set equal to $\frac{1}{N} = 0.2$.

Simulating over the interval $[0, 100]$ yields a total of 487 update times. Figure 6 depicts the time evolution of the difference between the largest and the smallest value of all the agents' states at each time instant that measures the deviation from consensus. As can be seen, this value can increase over some intervals of time, but over a large enough interval (related to the period over which the sequence of graphs is rooted and to the bound on the time delays) the overall difference decreases and tends to zero. Since this difference tends to zero, all states tend to the same value as illustrated in Figure 7.

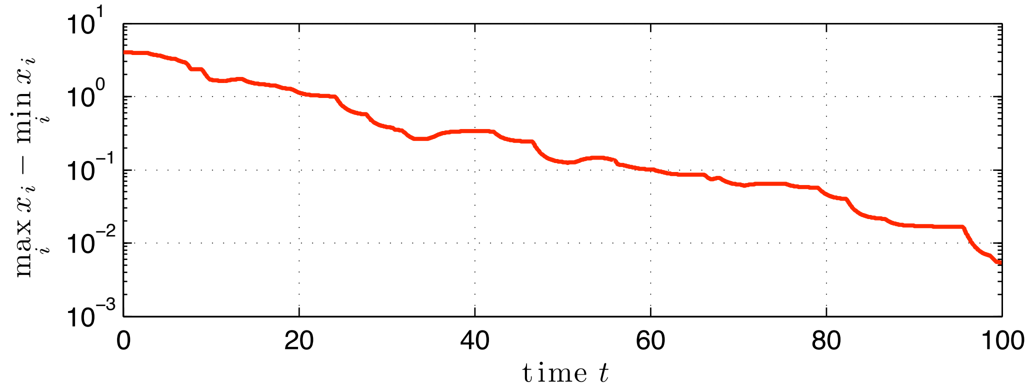


Figure 6: Time evolution of the deviation from consensus $\max_{i\in V} x_i - \min_{i\in V} x_i$.

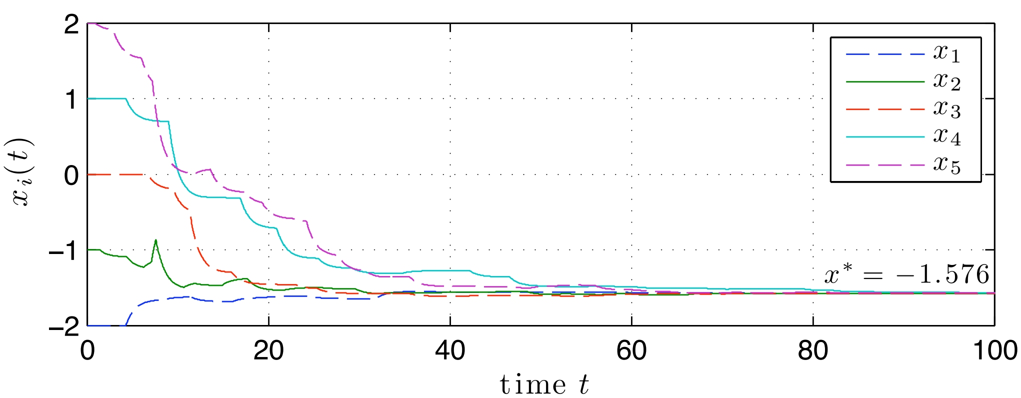


Figure 7: Time evolution of each agent's state x_i for $i \in \langle 1, N\rangle$.

5. Conclusion

The problem of consensus seeking was analyzed in the context of continuous-time variables with discrete-time updates. Besides the usual state variable for which consensus is sought, in our proposed solution an extra state variable is introduced for each agent. Between update times, both the original state and the extra variable evolve continuously. At update times, the extra variable is updated using information (possibly outdated) received from other agents, while the agent's state keeps the same value. The evolution of the aggregated state of the system is equivalently described by an appropriately defined discrete-time system. In this setup, both continuous evolution and discrete updates are interpreted as two different types of iterations. Time delays were incorporated by extending the set of update times and performing at each new time instant an identity iteration. Convergence to consensus was then established by resorting to existing discrete-time consensus results.

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