

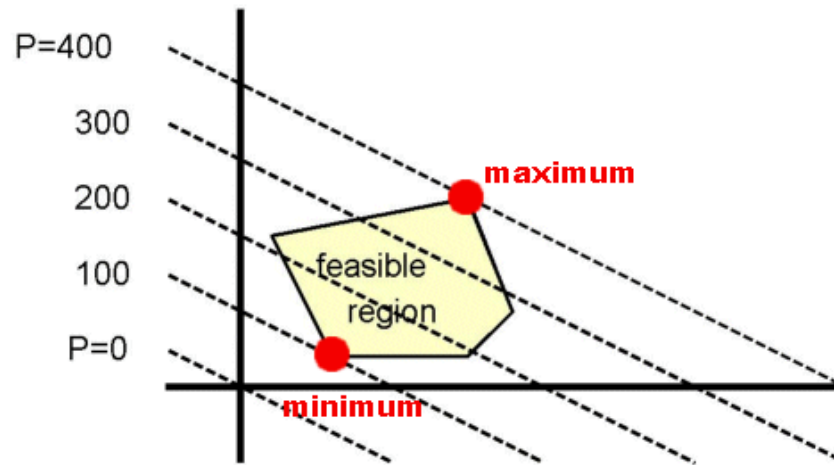
Distributed Real-Time Control Systems

Lecture 13-14

Distributed Control
Linear Programming

Linear Programs

- Optimize a linear function subject to a set of linear (affine) constraints.



- Many problems can be posed as linear programs and have wide application in operations research:
 - scheduling
 - assignment problems
 - supply chain management
 - automation
 - optimal pricing
 - transportation

Example: Mr. Silva's Electronics, Lda

- Mr. Silva has an electronics company that produces two products:
 - (P1) MP3 players
 - (P2) Alarm Clocks.
- MP3 players can be sold at **7€** and Alarm Clocks at **5€**.
- The company has a stock of **100 NIMH battery cells** and **240 push buttons**.
- Each **MP3 player** needs **2 battery cells** and **4 pushbuttons**.
- Each **Alarm Clock** needs **1 battery cell** and **3 pushbuttons**.
- **How many items of each product should Mr. Silva produce to maximize his profit ?**

Example: Mr. Silva's Electronics, Lda

x_1 : amount of MP3 players to produce

x_2 : amount of Alarm clocks to produce

•Maximize Profit:

$$7x_1 + 5x_2$$

•Battery Cells constraint:

$$2x_1 + 1x_2 \leq 100$$

•Push Buttons constraint:

$$4x_1 + 3x_2 \leq 240$$

•Positivity constraints:

$$x_1 \geq 0, \quad x_2 \geq 0$$

Example: Mr. Silva's Electronics, Lda

$$\begin{array}{ll} \text{maximize} & c^T x \\ \text{subject to} & Ax \leq b, \quad x \geq 0 \end{array}$$

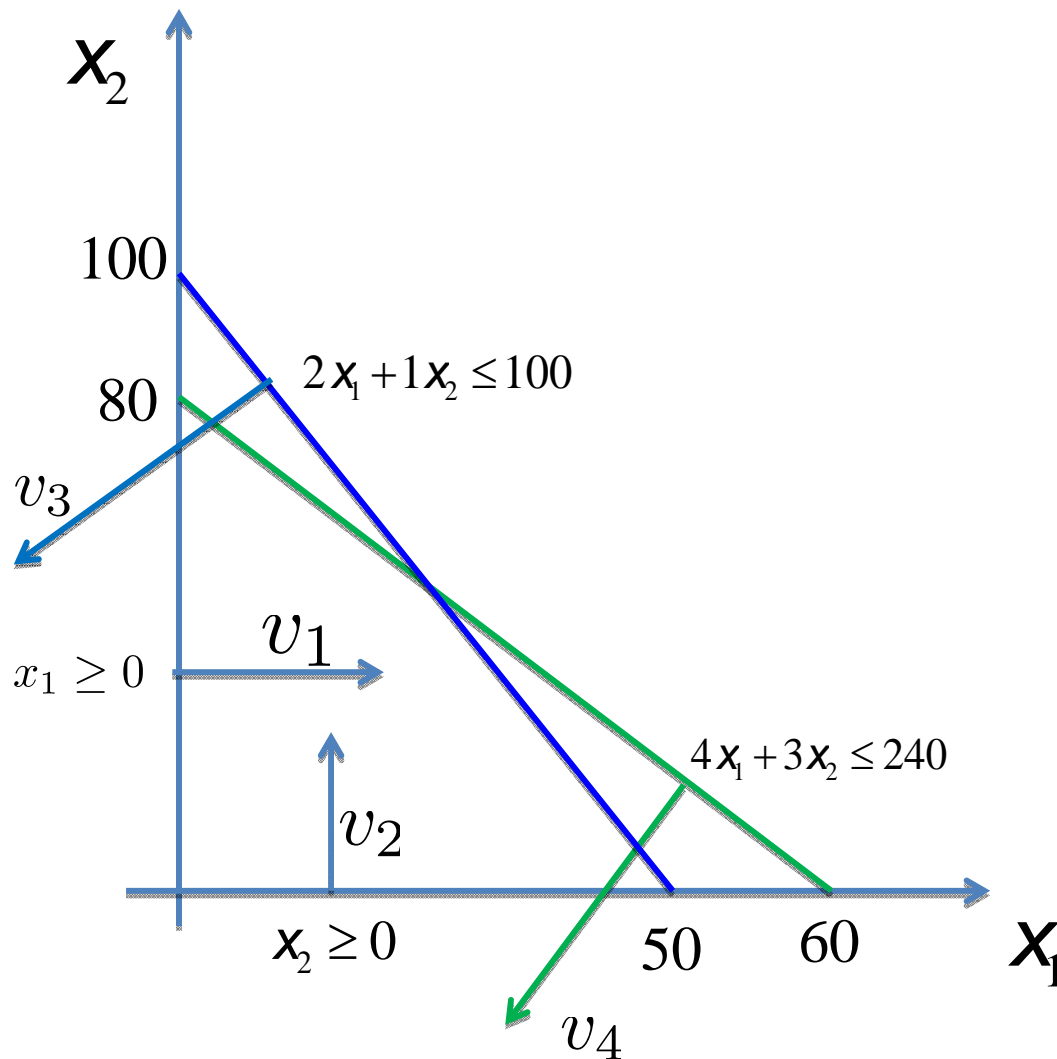
$$x = \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$$

$$c = \begin{bmatrix} 7 & 5 \end{bmatrix}$$

$$A = \begin{bmatrix} 2 & 1 \\ 4 & 3 \end{bmatrix}$$

$$b = \begin{bmatrix} 100 \\ 240 \end{bmatrix}$$

Example: Mr. Silva's Electronics, Lda



When constraints are written as:

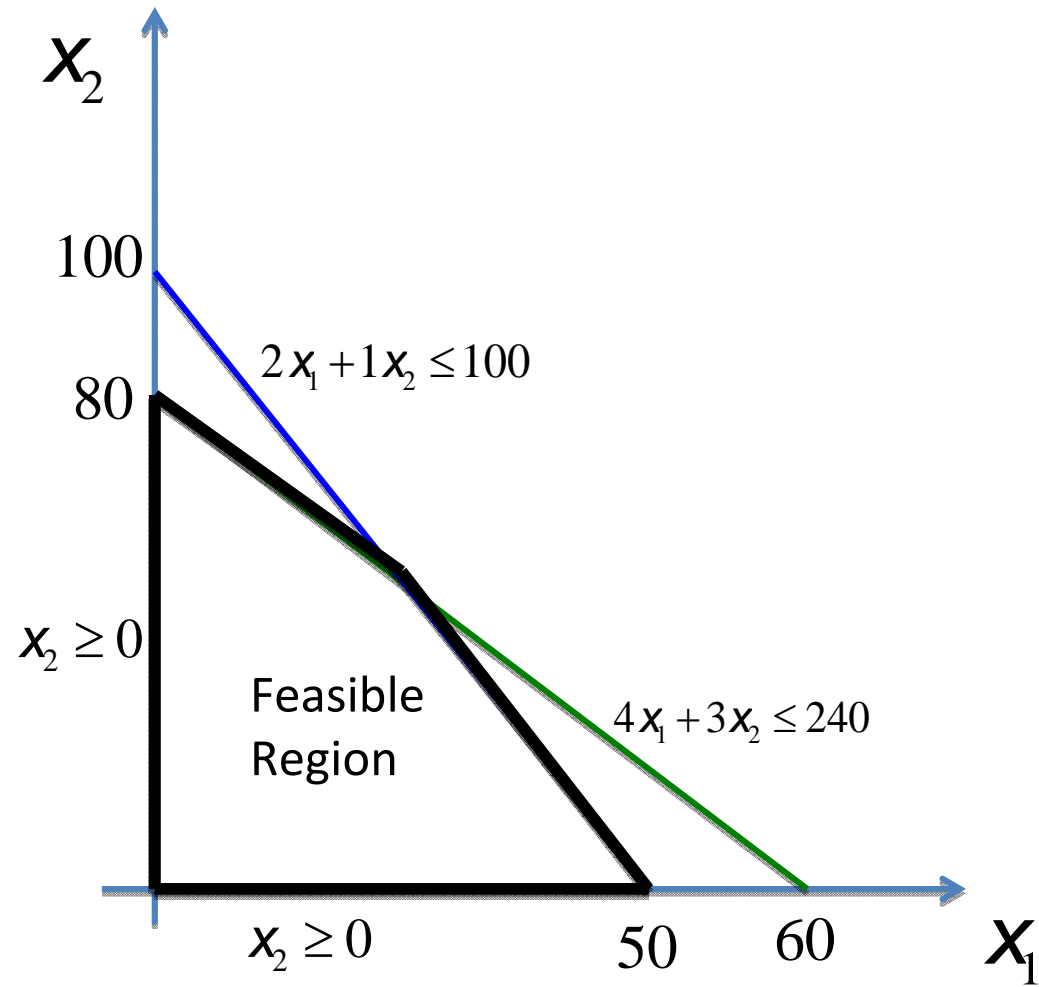
$$\mathbf{v}_i^T \mathbf{x} + b_i \geq 0$$

then, their gradient (\mathbf{v}_i), points toward the feasible region.

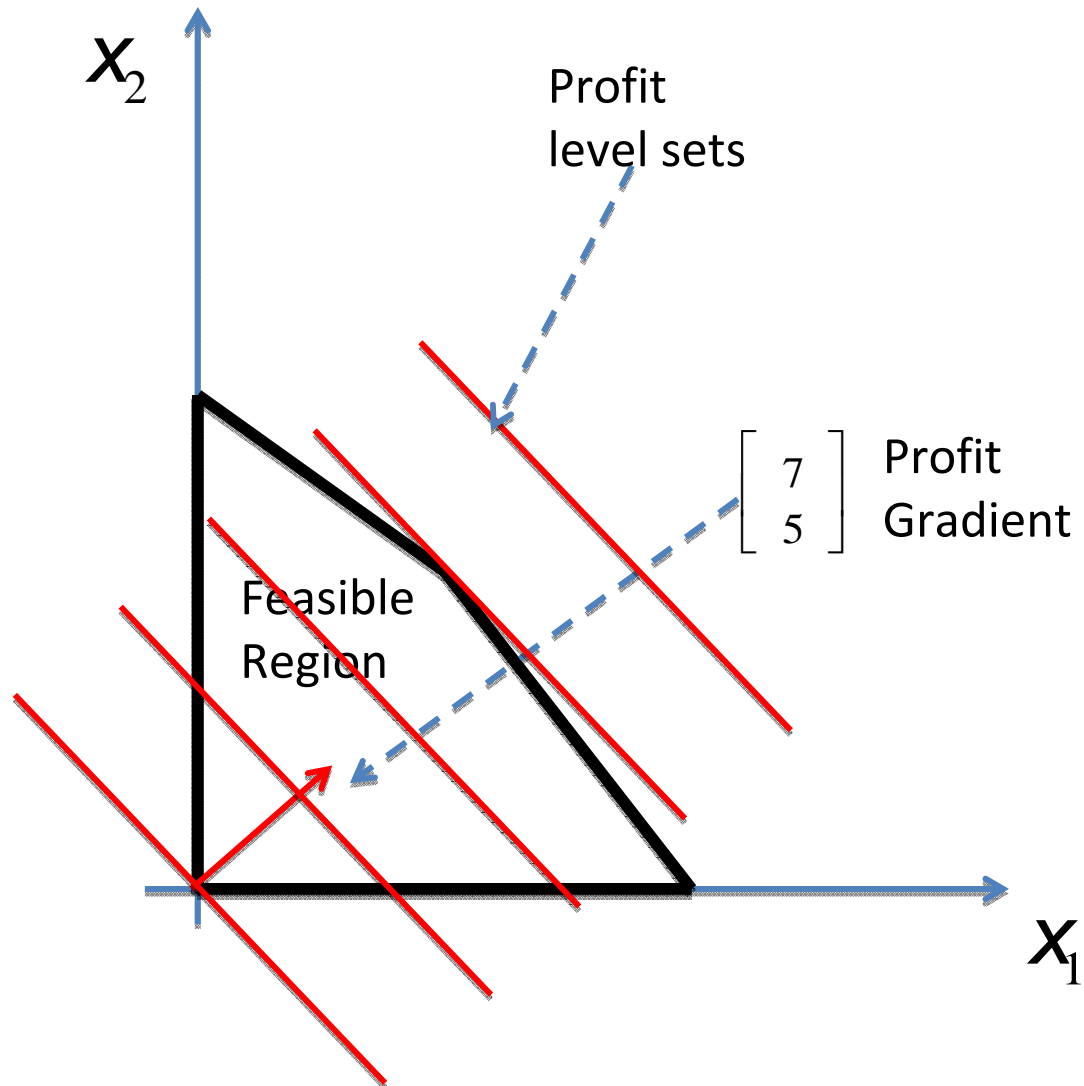
$$v_1 = \begin{bmatrix} 1 \\ 0 \end{bmatrix} \quad v_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}$$

$$v_3 = \begin{bmatrix} -2 \\ -1 \end{bmatrix} \quad v_4 = \begin{bmatrix} -4 \\ -3 \end{bmatrix}$$

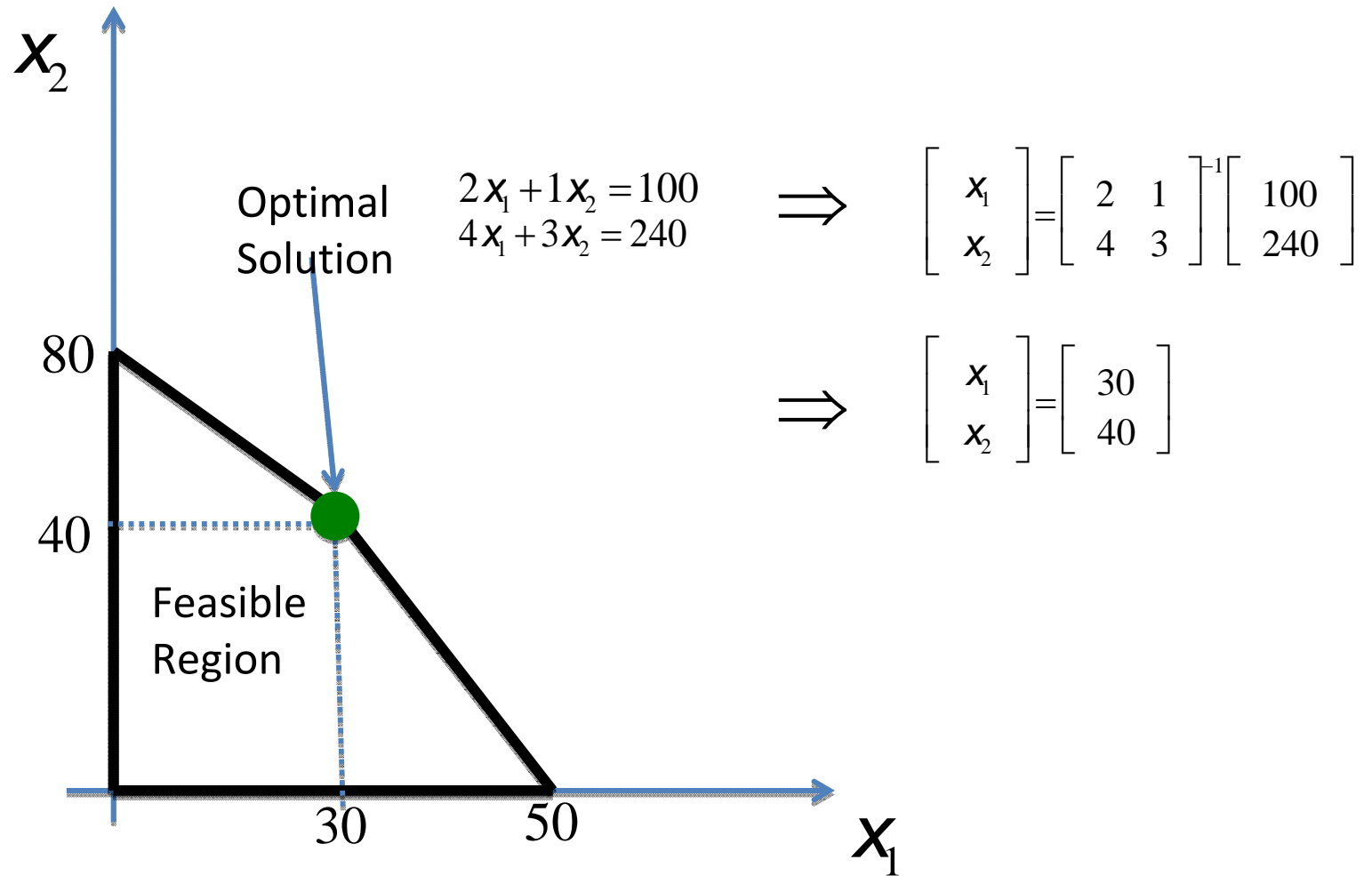
Example: Mr. Silva's Electronics, Lda



Example: Mr. Silva's Electronics, Lda



Example: Mr. Silva's Electronics, Lda

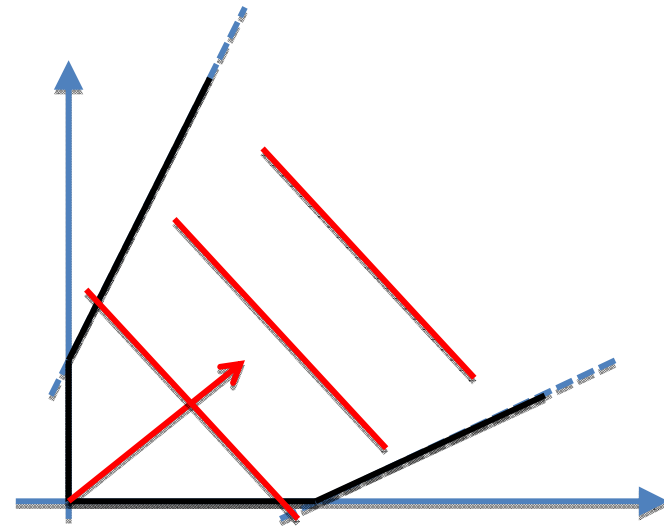


Properties of linear programs

- Feasible regions are convex polyhedra.
- Both minimization and maximization problems are convex (linear functions are both convex and concave).
- Local optima are also global optima.

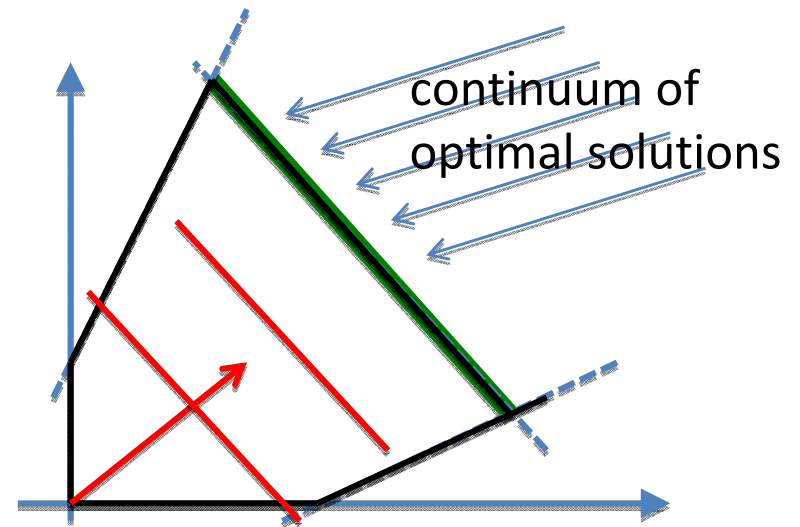
Impossible Cases

- **Unboundedness:**
 - If the feasible set is unbounded in the direction of the gradient, there is no finite solution (solution at infinity)
- **Infeasibility:**
 - If the conjunction of the constraint is the empty set. The problem is impossible (no solution)

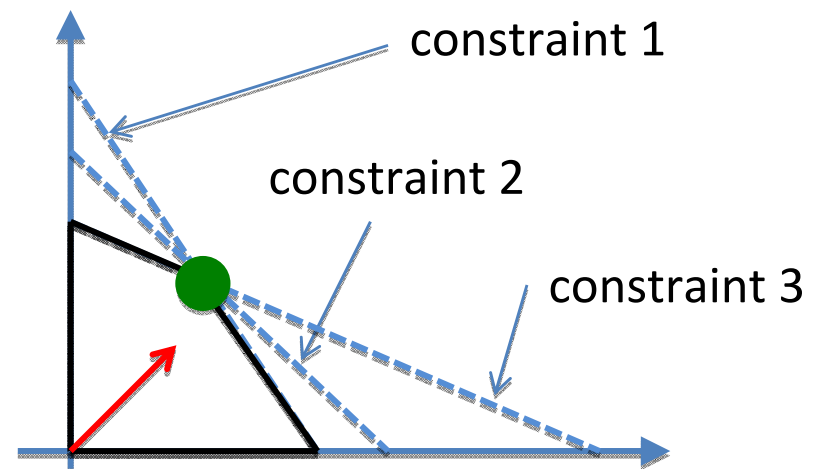


Degenerate Cases

- **Non-uniqueness:**
 - If the gradient is orthogonal to a constraint with optimal solutions, all “corners” associated to that constraint are also optima (multiple solutions)



- **Gauge freedom:**
 - Multiple “corners” at the same point. Solution is unique but some algorithms may have problems with this.

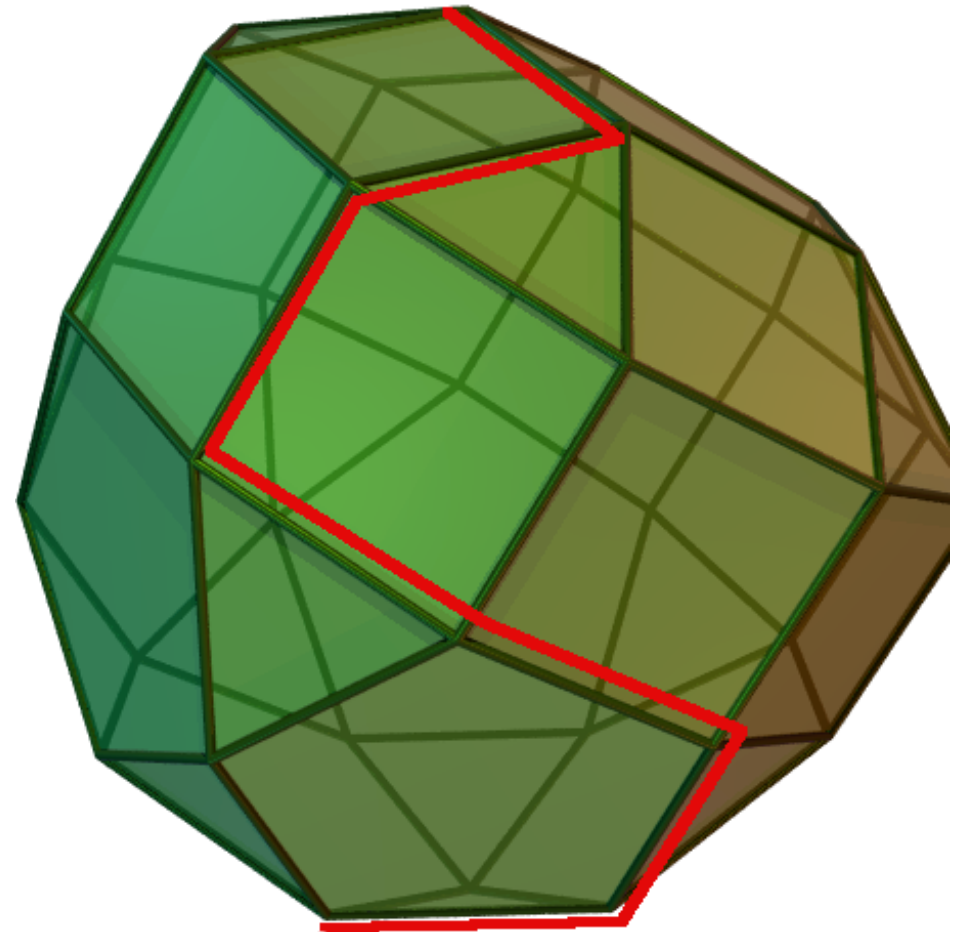


Solutions to Linear Programs

- Unfortunately most real-world problems have more than two variables. Graphical methods are not possible.
- Two main types of approaches:
 - Simplex methods: iterate over sequences of extreme points (vertices) of the feasibility polyhedron.
 - Interior point methods: iterate on the interior of the feasibility region.
- Performance depends on problem type.

Simplex methods

- Start on a feasible solution (vertex) of the problem and move along edges to a neighbor with a better cost.
- Stop when minimum is reached or unbounded edge visited.
- NOTE: Requires an initial feasible solution at a vertex.



Standard equality form

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \quad x \geq 0, \quad b \geq 0 \end{array}$$

- Minimization instead of maximization.
- Only equality constraints.
- All variables non-negative.
- All entries of vector b non-negative.

Conversion to Standard Equality Form

- If problem is “*max z*” convert it to “*min -z*”.

$$\begin{array}{c} \max \quad 7x_1 + 5x_2 \\ \downarrow \\ \min \quad -7x_1 - 5x_2 \end{array}$$

- If there are constraints with negative right hand side, multiply them by -1

$$\begin{array}{c} x_1 - 2x_2 \leq -8 \\ \downarrow \\ -x_1 + 2x_2 \geq 8 \end{array}$$

- To convert a unconstrained variables “*x*”, replace it by two new non-negative variables “*x⁺-x⁻*”.

$$\begin{array}{c} x_1 \quad \text{unrestricted} \\ \downarrow \\ x_1 = x_1^+ - x_1^-, \quad x_1^+ \geq 0, \quad x_1^- \geq 0 \end{array}$$

Conversion to Standard Equality Form

- Convert “lower-than-or-equal” inequalities to equalities by adding non-negative **slack** variables.
- Convert “greater-than-or-equal” inequalities to equalities by subtracting non-negative **surplus** variables.
- Set each slack and surplus variables’s coefficients in the cost function to zero.

$$4x_1 + 3x_2 \leq 240$$

↓

$$4x_1 + 3x_2 + s_1 = 240, \quad s_1 \geq 0$$

$$-x_1 + 2x_2 \geq 8$$

↓

$$-x_1 + 2x_2 - s_2 = 8, \quad s_2 \geq 0$$

Example: Mr. Silva's Electronics, Lda

x_1 : amount of MP3 players to produce

x_2 : amount of Alarm clocks to produce

s_1 : amount of unused battery cells.

s_2 : amount of unused push buttons.

• Minimize negative profit:

$$-7x_1 - 5x_2$$

• Battery Cells constraint:

$$2x_1 + x_2 + s_1 = 100$$

• Push Buttons constraint:

$$4x_1 + 3x_2 + s_2 = 240$$

• Positivity constraints:

$$x_1 \geq 0, \quad x_2 \geq 0, \quad s_1 \geq 0, \quad s_2 \geq 0$$

Example: Mr. Silva's Electronics, Lda

Linear Program in Standard Equality form:

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \quad x \geq 0, \quad b \geq 0 \end{array}$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ s_1 \\ s_2 \end{bmatrix} \quad c = \begin{bmatrix} -7 & -5 & 0 & 0 \end{bmatrix}$$
$$A = \begin{bmatrix} 2 & 1 & 1 & 0 \\ 4 & 3 & 0 & 1 \end{bmatrix} \quad b = \begin{bmatrix} 100 \\ 240 \end{bmatrix}$$

Definitions

$$\begin{array}{ll} \text{minimize} & c^T x \\ \text{subject to} & Ax = b, \quad x \geq 0 \end{array}$$

$$x \in R^n \quad A \in R^{d \times n}$$

$$b \in R^d \quad c \in R^n$$

$$c_i \in R - \text{ } i^{\text{th}} \text{ entry of vector } c$$

$$A_i \in R^d - \text{ } i^{\text{th}} \text{ column of matrix } A$$

$$h_i = \begin{bmatrix} c_i & A_i^T \end{bmatrix}^T - \text{ problem column}$$

$$H = \{h_i\}_{i=1 \dots n} - \text{ set of all problem columns}$$

$$G \subset H - \text{ A subset of the problem columns}$$

$$c_G - \text{ cost vector reconstructed from columns in } G$$

$$A_G - \text{ constraint matrix reconstructed from columns in } G$$

$$x_G - \text{ decision variables reconstructed from columns in } G$$

d = number of affine constraints

= number of rows of A

= number of slack/surplus variables

n = number of original + slack/surplus variables

Table Form

$$\begin{bmatrix} H & 0 \\ & b \end{bmatrix}$$

Fully represents the LP
in standard equality form

	x_1	x_2	s_1	s_2	
H	-7	-5	0	0	0
A	2	1	1	0	100
b	4	3	0	1	240

Basis

- Assume $\text{rank}(A)=d$. A set B of exactly d columns is called a basis if $\text{rank}(A_B)=d$
- Example. $B = \{h_3, h_4\}$

The diagram shows a matrix with 3 rows and 5 columns. The columns are labeled h_3 and h_4 above the third and fourth columns, respectively. A blue arrow points to the third column, and a red arrow points to the fourth column. A green box highlights the submatrix formed by the last two rows and the last two columns, which is labeled A_B with a green arrow pointing to it.

$$\begin{bmatrix} -7 & -5 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 100 \\ 4 & 3 & 0 & 1 & 240 \end{bmatrix}$$

Basic Solutions

- Basic solution is a solution x composed of d basic variables $x_B \neq 0$, and a set of $n-d$ non-basic variables $x_N=0$.
- A basic variable is a variable with only non-negativity constraints.
- A basic solution is feasible if all basic variables are positive.

• Example: $x = [0 \ 0 \ 100 \ 240]$

– Basic variables:

- $s_1 = 100$
- $s_2 = 240$

– Non-basic variables:

- $x_1 = 0$
- $x_2 = 0$

$$\begin{bmatrix} -7 & -5 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 100 \\ 4 & 3 & 0 & 1 & 240 \end{bmatrix}$$

- This is a trivial basic solution and can be used to initialize the simplex search.

Computing Basic Solutions

- Having selected a basis B , a basic solution can be computed by solving the linear system:

$$\begin{cases} A_B x_B = b \\ x_N = 0 \end{cases} \Rightarrow \begin{cases} x_B = A_B^{-1} b \\ x_N = 0 \end{cases}$$

- The associated cost is: $z_B = c_B^T x_B$
- The solution is feasible if all elements of x_B are non-negative.

Example: Mr. Silva's Electronics, Lda

Computing a non-trivial basic solution.

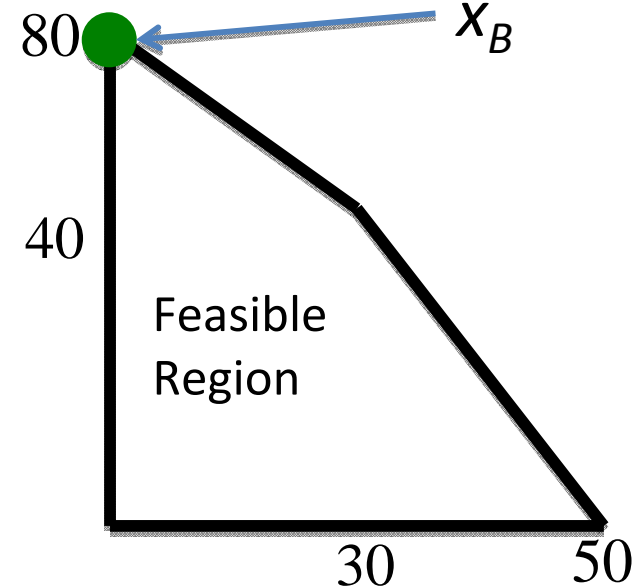
Let the basis be $B = \{h_2, h_3\}$.

$$\begin{bmatrix} -7 & \boxed{-5} & 0 & 0 & 0 \\ 2 & \boxed{1} & 1 & 0 & 100 \\ 4 & \boxed{3} & 0 & 1 & 240 \end{bmatrix}^b$$

c_B (blue arrow pointing to -5)
 A_B (red arrow pointing to the submatrix $\begin{bmatrix} 1 & 1 \\ 3 & 0 \end{bmatrix}$)

$$A_B^{-1} = \begin{bmatrix} 0 & 1/3 \\ 1 & -1/3 \end{bmatrix}$$

$$x_B = [x_2, s_1]^T = A_B^{-1}b = \begin{bmatrix} 80 & 20 \end{bmatrix}^T$$



FEASIBLE BASIC SOLUTION

Example: Mr. Silva's Electronics, Lda

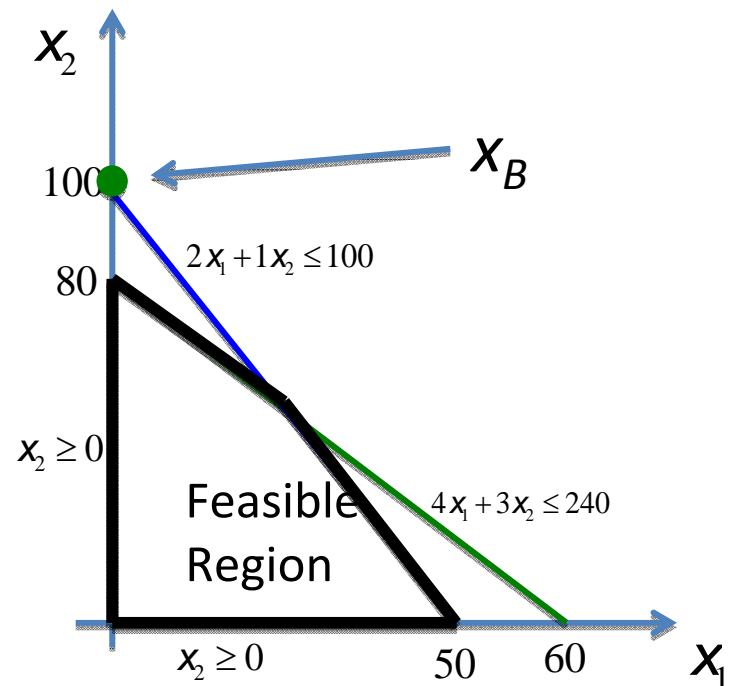
Now let the basis be $B = \{h_2, h_4\}$.

$$\left[\begin{array}{cccc|c} -7 & -5 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 100 \\ 4 & 3 & 0 & 1 & 240 \end{array} \right] b$$

c_B (blue arrows pointing to -5 and 0)
 A_B (red arrows pointing to 1 and 1)
 The right-hand side vector b is shown in a purple box.

$$A_b^{-1} = \begin{bmatrix} 1 & 0 \\ -3 & 1 \end{bmatrix}$$

$$x_B = [x_2, s_2]^T = [100, -60]^T$$



NON FEASIBLE BASIC SOLUTION

Adjacent Bases

- Two basis B_1 and B_2 are said to be adjacent if only one variable differs from one to the other, i.e. there exists an entering column “ e ” and a leaving column “ l ” such that:

$$e \in B_2, \quad l \in B_1, \quad B_2 = \{B_1 \cup e\} \setminus \{l\}$$

- Example: B_1 and B_2 are adjacent bases: $e = h_2, l = h_4$.

$$\begin{bmatrix} -7 & -5 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 100 \\ 4 & 3 & 0 & 1 & 240 \end{bmatrix}$$

Perturbation Analysis

- To evaluate the effect of changing from a basis to an adjacent one, let us consider:
 - B – current basis
 - x_B – current basic solution
 - x_h – non-basic variable, corresponding to column h .
- Let us perturb the solution by allowing x_h to have values above 0 (feasible). The basic variables will change their value to x'_B and the solution must verify:

$$\begin{array}{l} Ax = b \Rightarrow \\ \left[\begin{array}{cc} A_B & A_h \end{array} \right] \begin{bmatrix} x'_B \\ x_h \end{bmatrix} = b \\ x'_B = A_B^{-1} (b - A_h x_h) \end{array} \quad \longrightarrow \quad x'_B = x_B - A_B^{-1} A_h x_h$$

Example: Mr. Silva's Electronics, Lda

- Perturbed solution:

$$x'_B = x_B - A_B^{-1} A_h x_h$$

- Example: $B = \{h_2, h_3\}$
-

- $A_h = A_1, x_h = x_1$

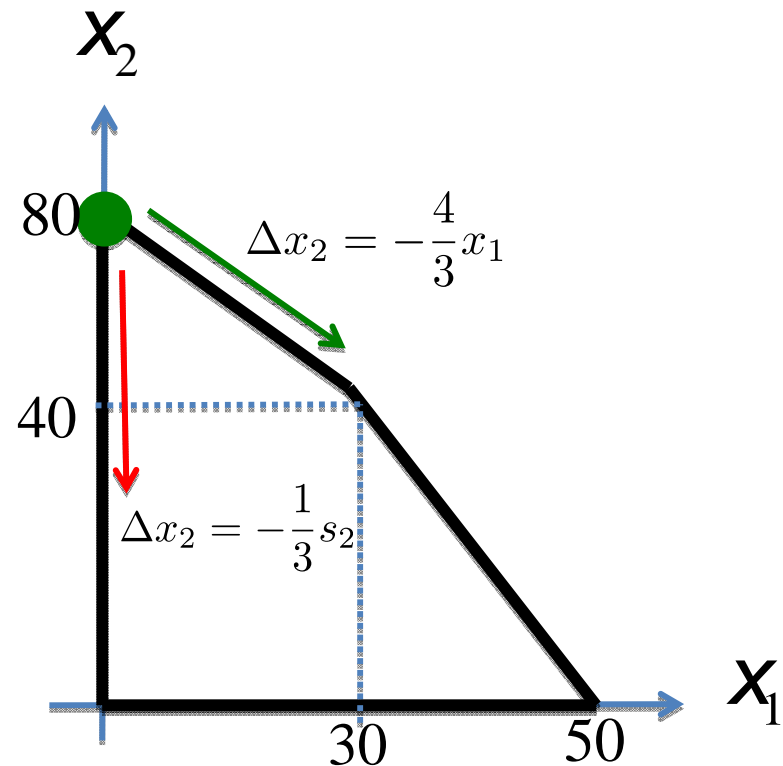
$$x'_B = \begin{bmatrix} 80 \\ 20 \end{bmatrix} - \begin{bmatrix} 0 & \frac{1}{3} \\ 1 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 2 \\ 4 \end{bmatrix} x_1$$
$$\begin{cases} x'_2 = 80 - \frac{4}{3}x_1 \\ s'_1 = 20 - \frac{2}{3}x_1 \end{cases}$$

- $A_h = A_4, x_h = s_2$

$$x'_B = \begin{bmatrix} 80 \\ 20 \end{bmatrix} - \begin{bmatrix} 0 & \frac{1}{3} \\ 1 & -\frac{1}{3} \end{bmatrix} \begin{bmatrix} 0 \\ 1 \end{bmatrix} x_1$$
$$\begin{cases} x'_2 = 80 - \frac{1}{3}s_2 \\ s'_1 = 20 + \frac{1}{3}s_2 \end{cases}$$

Graphical Analysis

- Perturbations around the current solution (green dot).
- A unit increase in the variable x_1 will make the solution travel along the green arrow and decrease $4/3$ on x_2 .
- A unit increase in variable s_2 will make the solution travel along the red arrow and decrease $1/3$ on x_2 .



Reduced cost

- Given a basis B and a single non-basic column h , its reduced cost represents its contribution to the cost (unitary cost variation) if it enters the basis:

$$z = c^T x = \begin{bmatrix} c_B^T & c_h \end{bmatrix} \begin{bmatrix} x'_B \\ x_h \end{bmatrix} \Rightarrow$$

$$z = c_B^T A_B^{-1} (b - A_h x_h) + c_h x_h$$

$$z = c_B^T x_B + (c_h - c_B^T A_B^{-1} A_h) x_h$$

$$\bar{c}_h = c_h - c_B^T A_B^{-1} A_h$$

What we gain by adding one element of this variable to the solution

What we lose by consuming the resources required.

Example: Mr. Silva's Electronics, Lda

- Reduced costs:

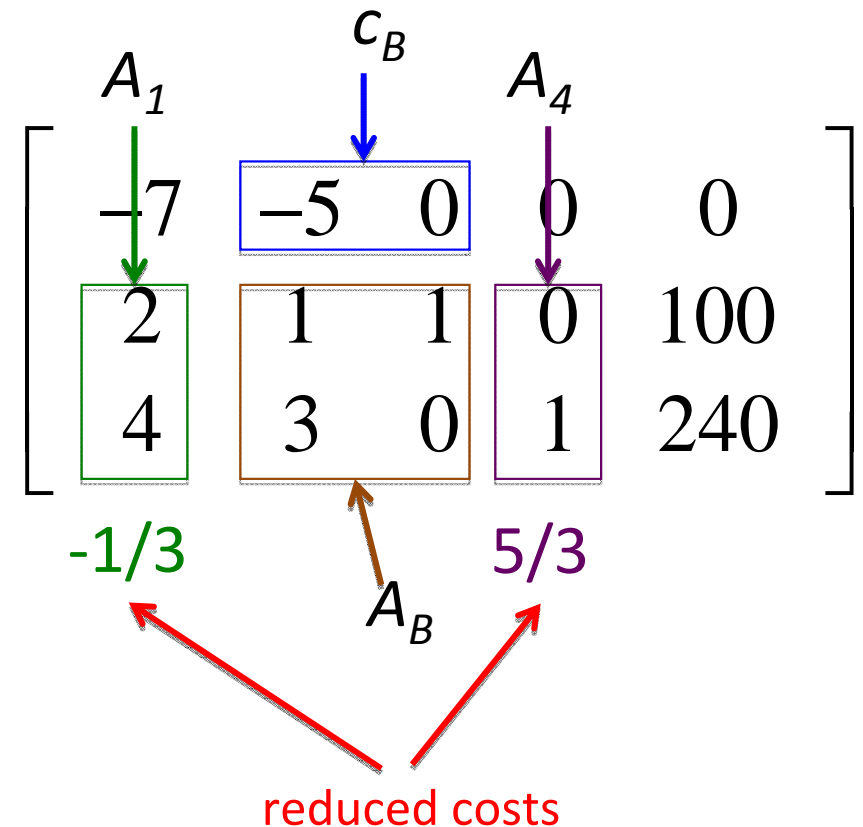
$$\bar{c}_h = c_h - c_B^T A_B^{-1} A_h$$

- Example: $B = \{h_2, h_3\}$

$$\bar{c}_h = c_h - A_h^T (A_B^{-1})^T c_B$$

$$\bar{c}_1 = c_1 - \begin{bmatrix} 2 & 4 \\ 1/3 & -1/3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -5 & 0 \end{bmatrix}$$

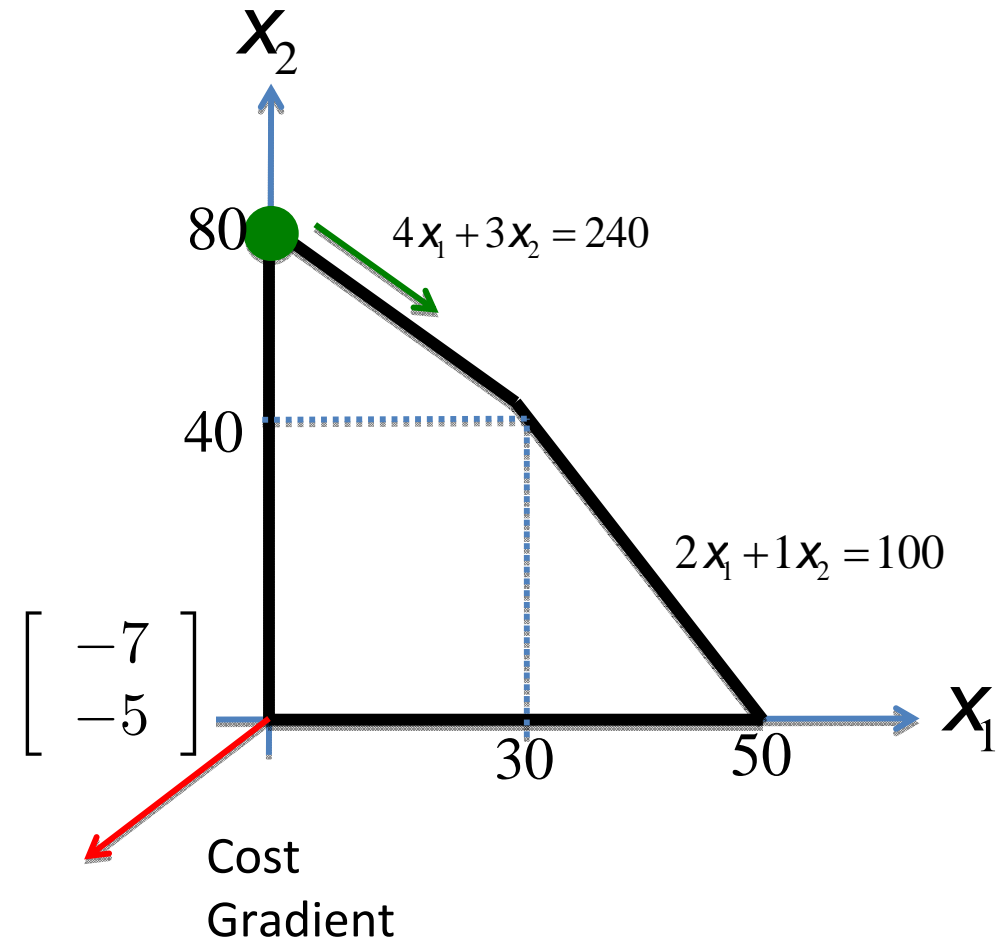
$$\bar{c}_4 = c_4 - \begin{bmatrix} 0 & 1 \\ 1/3 & -1/3 \end{bmatrix} \begin{bmatrix} 0 & 1 \\ -5 & 0 \end{bmatrix}$$



Graphical Analysis

- The cost variation is the inner product of the cost gradient with the perturbation vector.

$$\begin{aligned} \Delta c_1 &= \bar{c}_1 \Delta x_1 \\ &= \begin{bmatrix} -7 \\ -5 \\ 0 \\ 0 \end{bmatrix}^T \begin{bmatrix} 1 \\ -\frac{4}{3} \\ \frac{2}{3} \\ 0 \end{bmatrix} \Delta x_1 \\ &= -\frac{1}{3} \Delta x_1 \end{aligned}$$



Analysis

- Current solution:

$$x = \begin{bmatrix} 0 & 80 & 20 & 0 \end{bmatrix}$$

- Reduced costs:

$$\bar{c} = \begin{bmatrix} -1/3 & 0 & 0 & 5/3 \end{bmatrix}$$

A_1		A_4	
-7	-5	0	0
2	1	0	100
4	3	1	240
$-1/3$		$5/3$	

A_B

- Adding a unit of MP3 Player (x_1) has a value of 7 but in the current solution it requires 4 unavailable push buttons, that must be removed from alarm clocks. Because one alarm clock uses 3 push buttons, removing 4 push buttons will penalize us in 4/3 of the value of the alarm clock (5). Therefore we gain 7 but loose 20/3, so the net gain is 1/3.
- Adding a unit of a push button (s_2) has no value and forces us to remove 1 push button from an Alarm clock. By the same reasoning as before, we loose 1/3 of the value of an alarm clock, i.e. 5/3.

The entering column

- Any column whose reduced cost is negative can enter the basis:

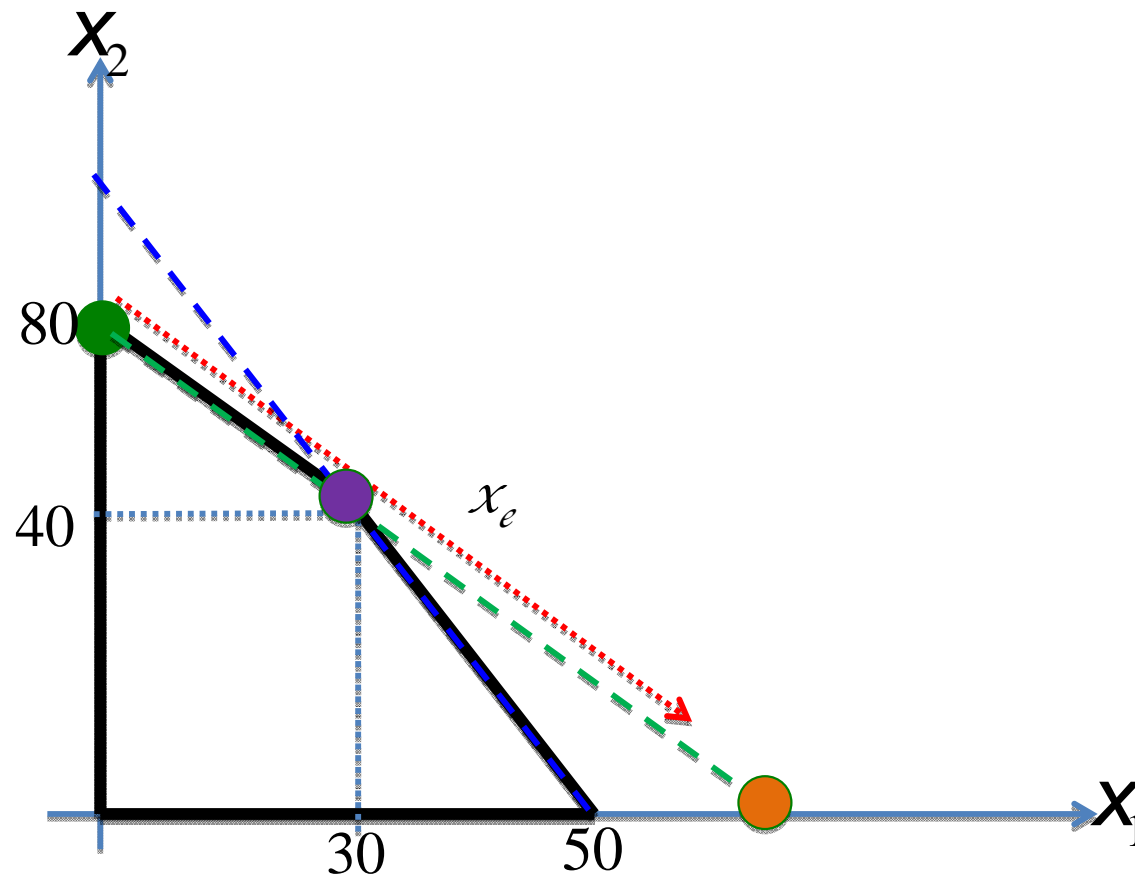
$$\bar{c}_h = c_h - A_h^T (A_B^{-1})^T c_B < 0$$

- Often it is chosen the column with the minimal negative cost:

$$e = \operatorname{argmin}_h \{ \bar{c}_h \mid \bar{c}_h < 0 \}$$

The leaving column

- Given an entering column “e” we are allowing the non-basic variable x_e to grow from zero to a positive value.
- As x_e grows, what is the first basic variable reaching value 0?



The leaving column

- The basic variables will change as (check the “Perturbation Analysis” slide):

$$\begin{bmatrix} A_B & A_e \end{bmatrix} \begin{bmatrix} x'_B \\ x_e \end{bmatrix} = b$$
$$x'_B = A_B^{-1} (b - A_e x_e)$$

The leaving column 2/2

- To assure feasibility we need:

$$\begin{aligned}x'_B &= A_B^{-1}(b - A_e x_e) \geq 0 \\(x'_B)_j &= (A_B^{-1}b)_j - (A_B^{-1}A_e)_j x_e \geq 0, \quad \forall j \\x_e &\leq \frac{(A_B^{-1}b)_j}{(A_B^{-1}A_e)_j}, \quad (A_B^{-1}A_e)_j > 0, \quad \forall j\end{aligned}$$

- The first variable x_j reaching 0 as x_e grows enters the non basic variable set.

$$j = \operatorname{argmin} \left\{ \frac{(A_B^{-1}b)_j}{(A_B^{-1}A_e)_j} \mid (A_B^{-1}A_e)_j > 0 \right\}$$

Optimal Solutions

- Fundamental theorems:
 - If an LP has an optimal feasible solution, then it has an optimal feasible **basic** solution.
 - Every fully non-degenerate LP has **at most one** optimal solution.
 - If an LP has an optimal solution and is non-degenerate, then there exists a **sequence of adjacent bases** from any basis B to the unique optimal basis B^* .

Basic Simplex Procedure

1. Let a **feasible basis B** be given.
2. While there exists an entering column “ e ” such that
$$e \notin B, \quad \bar{c}_e < 0$$
find a leaving basic column “ l ” such that
$$\{B \cup e\} \setminus \{l\}$$
is still a feasible basis.
3. Exchange “ l ” with “ e ” to find a **new basis**. This operation is denoted “Pivot”.

The initial basis

- Choose a subset of the columns of A forming an diagonal matrix (easily invertible), arising from slack variables.
- For constraints without slack variables, add an artificial variable with large associated cost M .
- Advantage: Initial basis is feasible.

Initial Basis : Artificial Variables

- Artificial variables allow an initial identity basis.
- Greater-than-or-equal constraints take surplus variables instead of slack variables.

$$2x_1 + x_2 \geq 100 \quad \Longrightarrow \quad 2x_1 + x_2 - s_1 + a_1 = 100, \quad s_1, a_1 \geq 0$$

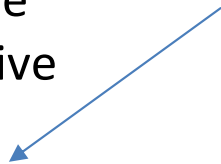
- Equality constraints have neither slack nor surplus variables.

$$4x_1 + 3x_2 = 240 \quad \Longrightarrow \quad 4x_1 + 3x_2 + a_2 = 240, \quad a_2 \geq 0$$

Initial Basis : Example

$$\begin{array}{l} \min -7x_1 - 5x_2 \\ \text{s.t.} \quad \begin{cases} 2x_1 + x_2 \geq 100 \\ -4x_1 - 3x_2 \geq 240 \end{cases} \end{array}$$

b_i 's must be non-negative



$$\begin{cases} 2x_1 + x_2 \geq 100 \\ -4x_1 - 3x_2 \geq 240 \end{cases} \longrightarrow \begin{cases} 2x_1 + x_2 - s_1 + a_1 = 100 \\ -4x_1 - 3x_2 - s_2 + a_2 = 240 \end{cases}$$

x_1	x_2	s_1	s_2	a_1	a_2	b	
-7	-5	0	0	M	M		
2	1	-1	0	1	0	100	<i>Initial basis</i>
-4	-3	0	-1	0	1	240	



The iterations

- As seen in the previous slides the inverse of the basis matrix A_B plays an important role in many operations.

- Computing the basic solution

$$x_B = A_B^{-1} b$$

- Computing the reduced cost

$$\bar{c}_h = c_h - c_B^T A_B^{-1} A_h$$

- Select the entering column

$$e = \operatorname{argmin}_h \{ \bar{c}_h \mid \bar{c}_h < 0 \}$$

- Selecting the leaving column

$$j = \operatorname{argmin} \left\{ \left(A_B^{-1} b \right)_j / \left(A_B^{-1} A_e \right)_j \mid \left(A_B^{-1} A_e \right)_j > 0 \right\}$$

Gauss Elimination

- When the initial basis is the identity, the computation of the inverse can be done in a simple way with Gaussian elimination at each pivot operation, and the problem table transformed so as the new basis is the identity again.

$$[A_B \ A_N] \begin{bmatrix} x_B \\ x_N \end{bmatrix} = b \quad \xrightarrow{\times A_B^{-1}} \quad [I \ A_B^{-1} A_N] \begin{bmatrix} x_B \\ x_N \end{bmatrix} = A_B^{-1} b$$

- Having the basis as the identity is very useful because the solution, entering and leaving columns can be checked by inspection.

Gauss Elimination

- Remember Mr. Silva.

$$\left[\begin{array}{cccc|c} \text{\scriptsize } c^T & & & & \\ \hline -7 & -5 & 0 & 0 & 0 \\ \hline 2 & 1 & 1 & 0 & 100 \\ \hline 4 & 3 & 0 & 1 & 240 \\ \hline \end{array} \right]$$

A b

$c_B^T x_B$

Gauss Elimination

- Starting with identity basis...

$$\begin{bmatrix} x_1 & x_2 & s_1 & s_2 & b \\ -7 & -5 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 100 \\ 4 & 3 & 0 & 1 & 240 \end{bmatrix}$$


 A_B

Gauss Elimination

- We want to make columns 1 enter the basis and column 3 leave the basis.

$$\left[\begin{array}{cc|cc|c} x_1 & x_2 & s_1 & s_2 & b \\ -7 & -5 & 0 & 0 & 0 \\ \hline 2 & 1 & 1 & 0 & 100 \\ 4 & 3 & 0 & 1 & 240 \end{array} \right]$$

The diagram shows a simplex tableau with columns for variables x_1 , x_2 , slack variables s_1 , s_2 , and the right-hand side b . The second row is the objective function row. The third and fourth rows are constraint rows. A green box highlights the pivot element '2' in the second row, first column, with a green arrow labeled 'e' pointing to it. A blue box highlights the element '1' in the third row, first column, with a blue arrow labeled 'l' pointing to it. A red box highlights the entire third row, with a red arrow labeled A_B pointing to it.

Gauss Elimination

- The new basis will be composed by columns 1 and 4.

$$\begin{bmatrix}
 x_1 & x_2 & s_1 & s_2 & b \\
 -7 & -5 & 0 & 0 & 0 \\
 2 & 1 & 1 & 0 & 100 \\
 4 & 3 & 0 & 1 & 240
 \end{bmatrix}$$

$A_{B'}$
 A_B

- We have to make linear combinations of the rows of $[A \ b]$ to let the identity matrix “emerge” from columns 1 and 4.

Gauss Elimination

- Locate the pivot element a_{je} in the entering column e at the row where the leaving column had value 1.

$$\begin{bmatrix} x_1 & x_2 & s_1 & s_2 & b \\ -7 & -5 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 100 \\ 4 & 3 & 0 & 1 & 240 \end{bmatrix}$$

Pivot $A_{B'}$

- The objective is to make the pivot equal to 1 and all other elements of that column equal to 0

Gauss Elimination

- Divide the pivot line a_j by the pivot value a_{je} .

$$a_j \leftarrow a_j / a_{je}$$

Pivot →


$$\begin{bmatrix} x_1 & x_2 & s_1 & s_2 & b \\ -7 & -5 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 100 \\ 4 & 3 & 0 & 1 & 240 \end{bmatrix}$$

$A_{B'}$

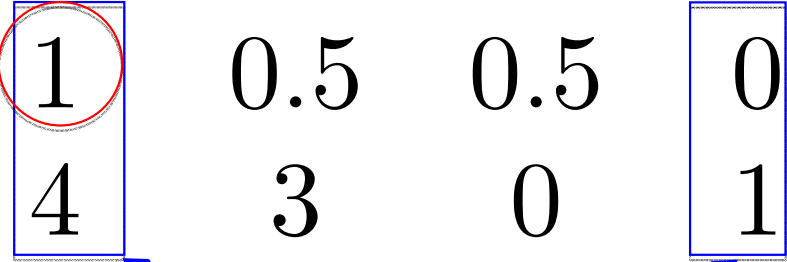
Gauss Elimination

- Divide the pivot line a_j by the pivot value a_{je} .

$$a_j \leftarrow a_j / a_{je}$$

Pivot 

$$\begin{bmatrix} x_1 & x_2 & s_1 & s_2 & b \\ -7 & -5 & 0 & 0 & 0 \\ 1 & 0.5 & 0.5 & 0 & 50 \\ 4 & 3 & 0 & 1 & 240 \end{bmatrix}$$




$A_{B'}$

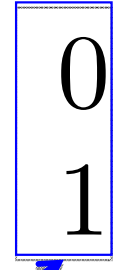
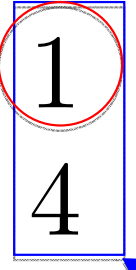
Gauss Elimination

- Let the remaining lines be $a_i = a_i - a_{ie}a_j$

$$a_i \leftarrow a_i - a_{ie}a_j$$

Pivot 

$$\begin{bmatrix} x_1 & x_2 & s_1 & s_2 & b \\ -7 & -5 & 0 & 0 & 0 \\ 1 & 0.5 & 0.5 & 0 & 50 \\ 4 & 3 & 0 & 1 & 240 \end{bmatrix}$$



$A_{B'}$

Gauss Elimination

- Let the remaining lines be $a_i = a_i - a_{ie}a_j$

$$a_i \leftarrow a_i - a_{ie}a_j$$

Pivot →

x_1	x_2	s_1	s_2	b
-7	-5	0	0	0
1	0.5	0.5	0	50
0	1	-2	1	40

$A_{B'}$

Gauss Elimination

- Let the remaining lines be $a_i = a_i - a_{ie}a_j$

$$a_i \leftarrow a_i - a_{ie}a_j$$

Pivot →

$$\left[\begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & b \\ 0 & -1.5 & 3.5 & 0 & 350 \\ 1 & 0.5 & 0.5 & 0 & 50 \\ 0 & 1 & -2 & 1 & 40 \end{array} \right]$$

$A_{B'}$

Gauss Elimination

- The new basic solution is now $x_1 = 50, s_2 = 40$
- The associated cost is 350

Pivot

$$\left[\begin{array}{ccccc} x_1 & x_2 & s_1 & s_2 & b \\ 0 & -1.5 & 3.5 & 0 & 350 \\ 1 & 0.5 & 0.5 & 0 & 50 \\ 0 & 1 & -2 & 1 & 40 \end{array} \right]$$

$A_{B'}$

Simple computations

- Once the basis is the identity, computations are simpler:

$$x_B = b'$$

$$\bar{c}_h = c_h - c_B^T A'_h$$

$$e = \operatorname{argmin}_h \{ \bar{c}_h \mid \bar{c}_h < 0 \}$$

$$j = \operatorname{argmin} \{ (b')_j / (A'_e)_j \mid (A'_e)_j > 0 \}$$

The Simplex Algorithm (1/2)

1. Put the problem in standard equality form for minimization with enough slack/auxiliary variables to have an initial identity basis.
2. Determine which variable to enter the solution. Identify the column with the lowest negative reduced cost. This is the **pivot column e** .

$$e = \operatorname{argmin}_h \{ \bar{c}_h \mid \bar{c}_h < 0 \}$$

3. Determine which variable to replace. Consider only rows with positive values in the pivot column. Divide each amount in the quantity column (last column) by the corresponding number of the entering column. The row with the smallest non-negative number is the **pivot row j** . If all are negative, the problem is unbounded. Exit.

$$j = \operatorname{argmin} \{ (b')_j / (A'_e)_j \mid (A'_e)_j > 0 \}$$

The Simplex Algorithm (2/2)

4. Compute new values for the pivot row. Divide all elements by the pivot number.

$$a_j \leftarrow a_j / a_{je}$$

5. Compute new values for the remaining rows. Take the element of this row at the pivot column, multiply it by the pivot row, and subtract the result from itself.

$$a_i \leftarrow a_i - a_{ie} a_j$$

6. Compute the reduced costs. If all are non-negative, the algorithm converged, goto 7. Otherwise goto 2.

$$\bar{c}_h = c_h - c_B^T A'_h$$

7. If an artificial variable is in the basis, the problem is unfeasible. Otherwise this is the optimal solution. Exit.

Mr. Silva Full Solution

- Step 1 – Table Form

$$\begin{bmatrix} -7 & -5 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 100 \\ 4 & 3 & 0 & 1 & 240 \end{bmatrix}$$

Mr. Silva Full Solution

- Step 2 – Compute reduced costs and choose entering column.

$$\begin{bmatrix} -7 & -5 & 0 & 0 & 0 \\ 2 & 1 & 1 & 0 & 100 \\ 4 & 3 & 0 & 1 & 240 \end{bmatrix}$$

$$\begin{matrix} -7 & -5 & 0 & 0 \end{matrix}$$

$$e = \underset{h}{\operatorname{argmin}} \{ c_h - A_h^T c_B \mid c_h < 0 \}$$

Mr. Silva Full Solution

- Step 3 – Determine which variable to replace

$$\begin{array}{ccccc|c}
 \boxed{-7} & -5 & \boxed{0} & 0 & 0 & \\
 \boxed{2} & 1 & \boxed{1} & 0 & 100 & \textcircled{50} \\
 \boxed{4} & 3 & \boxed{0} & 1 & 240 & 60
 \end{array}$$

$$\textcircled{-7} \quad -5 \quad 0 \quad 0$$

$$j = \operatorname{argmin} \left\{ (b')_{j} / (A'_e)_j \mid (A'_e)_j > 0 \right\}$$

Mr. Silva Full Solution

- Step 4 – Normalize Pivot Row

$$\left[\begin{array}{ccccc} -7 & -5 & 0 & 0 & 0 \\ 1 & 0.5 & 0.5 & 0 & 50 \\ 4 & 3 & 0 & 1 & 240 \end{array} \right] \begin{array}{l} 50 \\ 60 \end{array}$$

$$\begin{array}{cccc} -7 & -5 & 0 & 0 \end{array}$$

$$a_j \leftarrow a_j / a_{je}$$

Mr. Silva Full Solution

- Step 5 – Compute remaining rows

$$\left[\begin{array}{ccccc} -7 & -5 & 0 & 0 & 0 \\ 1 & 0.5 & 0.5 & 0 & 50 \\ 0 & 1 & -2 & 1 & 40 \end{array} \right] \begin{array}{l} 50 \\ 60 \end{array}$$

$$\begin{array}{cccc} -7 & -5 & 0 & 0 \end{array}$$

$$a_i \leftarrow a_i - a_{ie} a_j$$

Mr. Silva Full Solution

- Step 6 – Recompute reduced costs

$$\bar{c}_h = c_h - c_B^T A'_h$$

$$\begin{bmatrix} -7 & -5 & 0 & 0 & 0 \\ 1 & 0.5 & 0.5 & 0 & 50 \\ 0 & 1 & -2 & 1 & 40 \end{bmatrix}$$

$$0 \quad -1.5 \quad 3.5 \quad 0$$

- Goto 2.

Mr. Silva Full Solution

- Step 2 – Choose entering column

$$\begin{bmatrix} -7 & -5 & 0 & 0 & 0 \\ 1 & 0.5 & 0.5 & 0 & 50 \\ 0 & 1 & -2 & 1 & 40 \end{bmatrix}$$

$$0 \quad -1.5 \quad 3.5 \quad 0$$

$$e = \underset{h}{\operatorname{argmin}} \{ c_h - A_h^T c_B \mid c_h < 0 \}$$

Mr. Silva Full Solution

- Step 3 – Determine which variable to replace

$$\begin{array}{ccccc|c}
 -7 & -5 & 0 & 0 & 0 & 100 \\
 1 & 0.5 & 0.5 & 0 & 50 & \\
 0 & 1 & -2 & 1 & 40 & 40 \\
 \hline
 0 & -1.5 & 3.5 & 0 & &
 \end{array}$$

$$j = \operatorname{argmin} \left\{ (b')_j / (A'_e)_j \mid (A'_e)_j > 0 \right\}$$

Mr. Silva Full Solution

- Step 4 – Normalize Pivot Row

$$\left[\begin{array}{ccccc} -7 & -5 & 0 & 0 & 0 \\ 1 & 0.5 & 0.5 & 0 & 50 \\ 0 & 1 & -2 & 1 & 40 \\ 0 & -1.5 & 3.5 & 0 & 0 \end{array} \right] \begin{array}{l} 100 \\ 40 \end{array}$$

$$a_j \leftarrow a_j / a_{je}$$

Mr. Silva Full Solution

- Step 5 – Compute remaining rows

$$\left[\begin{array}{ccccc} -7 & -5 & 0 & 0 & 0 \\ 1 & 0 & 1.5 & -0.5 & 30 \\ 0 & 1 & -2 & 1 & 40 \end{array} \right] \begin{array}{l} 100 \\ 40 \end{array}$$

$$0 \quad -1.5 \quad 3.5 \quad 0$$

$$a_i \leftarrow a_i - a_{ie} a_j$$

Mr. Silva Full Solution

- Step 6 – Re-compute reduced costs

$$\bar{c}_h = c_h - c_B^T A'_h$$

$$\begin{bmatrix} -7 & -5 & 0 & 0 & 0 \\ 1 & 0 & 1.5 & -0.5 & 30 \\ 0 & 1 & -2 & 1 & 40 \end{bmatrix}$$

$$0 \quad 0 \quad 0.5 \quad 1.5$$

- All reduced costs positive: converged.

Mr. Silva Full Solution

- Step 7 – Conclude

$$\begin{bmatrix} -7 & -5 & 0 & 0 & 0 \\ 1 & 0 & 1.5 & -0.5 & 30 \\ 0 & 1 & -2 & 1 & 40 \end{bmatrix}$$

- No artificial variable in the basis.
 - Optimal solution: $x_1 = 30$, $x_2 = 40$, remaining 0.

Homework

Formulate the illumination control problem with two luminaires as a linear program with energy minimization. Solve the problem for a sensible set of values for the cost and coupling constraints.