Proof sketches

Asynchronous algorithms: analysis

Lemma A:

Let $\{x(k)\}_{k\in\mathbb{N}}$ sequence of points produced by the asynchronous algorithms I or II

then,

1) $\sum_{k\geq 1} \|\nabla \hat{f}\,(x(k))\,\|^2 < \infty, a.s.;$
2) $\nabla \hat{f}\,(x(k)) \to 0, a.s.$

Asynchronous algorithms: analysis

Lemma B:

Let $\{x(k)\}_{k\in\mathbb{N}}$ sequence generated according to Lemma A, with probability one

then,

$$\hat{f}(x(k)) \downarrow \hat{f}^{\star}$$

and there exists a subsequence converging to a point in the solution set:

$$x(k_l) \rightarrow y, y \in \mathcal{X}^{\star}$$

Proof of almost sure convergence

Suppose $\mathrm{d}_{\mathcal{X}^\star}(x(k)) \not\to 0$

Then, there is an $\epsilon > 0$ and a subsequence $\{x(k_l)\}_{l\in\mathbb{N}}$

such that $\mathrm{d}_{\mathcal{X}^\star}(x(k_l)) > \epsilon$

As the function is coercive, continuous, and convex, and whose gradient by Lemma A vanishes, then, by Lemma B there is a subsequence of $\{x(k_l)\}_{l\in\mathbb{N}}$ converging to a point in $\mathcal{X}^\star$

Sketch of proof for the almost sure convergence to a point

Fix an $x^\star \in \mathcal{X}^\star$

Firstly, we prove $\{\|x(k) - x^\star\|^2\}_{k\in\mathbb{N}}$ is convergent.

$$\mathbb{E}\left[\|x(k) - x^\star\|^2 | \mathcal{F}_{k-1}\right] = \sum_{i=1}^{n} \frac{1}{n} \left\| x(k-1) - \frac{1}{L_{\hat{f}}} g_i(k-1) - x^\star \right\|^2$$

$$\|x(k-1) - x^\star\|^2 + \frac{1}{nL_{\hat{f}}^2} \left\|\nabla \hat{f}(x(k-1))\right\|^2 - \frac{2}{nL_{\hat{f}}} (x(k-1) - x^\star)^\top \nabla \hat{f}(x(k-1))$$

Sketch of proof for the almost sure convergence to a point

$$(x(k-1)-x^\star)^\top \nabla \hat{f}(x(k-1)) = (x(k-1)-x^\star)^\top (\nabla \hat{f}(x(k-1)) - \nabla \hat{f}(x^\star)) \geq 0$$

$$\mathbb{E}\left[\|x(k)-x^\star\|^2 | \mathcal{F}_{k-1}\right] \leq \|x(k-1)-x^\star\|^2 + \frac{1}{nL_{\hat{f}}^2}\left\|\nabla \hat{f}(x(k-1))\right\|^2$$

As proved in Lemma A, the sum $\sum_{k\geq 1} \|\nabla \hat{f}(x(k))\|^2 < \infty, a.s.$

so we can invoke the result in Robbins,1985 to state the convergence of the squared distance.