Smooth Priorities for Multi-Product Inventory Control

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Abstract: Whenever dealing with periodic review multi-products inventory control for capacitated machines, one of the main issues that has to be addressed concerns the dynamic capacity allocation. That is, how to assign capacity to the several competing products that require more than it is available. One typical approach is to assign priorities to the products, according to some degree of relative importance. Whereas priority based capacity allocation is attractive, due to its simplicity and for the fact that it makes sense in a significant number of settings, we contend this to be too unfair to lower priority products, given their access to production tends to be highly variable and uncertain. Departing from what we call strict priorities, we propose a priority based mechanism that improves on this drawback, termed smooth priorities. This new policy for multi-product, limited capacity production systems with stochastic demand is studied. Theoretical comparisons are made to the common policies of strict priorities and linear scaling. An optimizer based on Infinitesimal Perturbation Analysis, IPA, simulation is devised and results of practical comparison between smooth and strict priorities are presented. The structure of the cost function with smooth priorities is studied through function plots obtained from simulation and numerical results show consistent better performances than those achieved under strict priorities.

Keywords: Periodic Review Inventory Control, Multiple products and machines, Capacitated Systems, Base-stock Echelon Policies, Infinitesimal Perturbation Analysis.

1. INTRODUCTION

In the context of multi-product, limited capacity production systems, with stochastic demand and periodic review, the optimal policy has defied closed form formulations and shown to be excessively complex for numerical calculation. For this reason, it is frequent to use sub-optimal policies characterized by a small set of parameters which may then allow for simulation based optimization. Within the sub-optimal policies normally used, the most common are base-stock policies. Whereas defining the base-stocks is enough for single product systems, being them composed of capacitated or uncapacitated machines, such is not the case for multiple product systems with capacitated machines. In those cases one has still to address the issue of dynamic capacity allocation when, for a given machine, the requests from all products exceed its capacity. Many are the schemes to allocate capacity dynamically, out of which decisions based on a priority list is one of the most common. However, a strict priority policy, while interesting in simplicity and for the high priority products, is not as interesting when it comes to satisfying the demand for lower priority products. There may be periods when a subset of the higher priority products exhausts all the available capacity for those periods, and the lower priority products will have to wait for a subsequent period to see their needs fully or partially satisfied. A consequence of this is the fact that their levels of safety stock tend to increase to hedge against a less reliable, with higher variance, access to the resources. Furthermore, determining the adequate list of priorities is a combinatorial problem of difficult resolution, when the number of different products is significant.

The objective of this paper is to introduce and study the smooth priorities concept, as an alternative to strict priorities. Smooth priorities try to overcome the unfairness of strict priorities by allowing a slice of the available capacity to all products in all periods, irrespective of being higher or lower priority. Being a priority inspired concept, the slice each product gets on a given period is also a function of a set of weights, which express the relative priorities among the products. Given the fact that those weights can be defined as real numbers between 0 and 1, transforms the problem of determining the relative importance of the products into a continuous variable, non-linear programming problem.

A great tutorial that describes the current state of the art in production systems is [Hou06]. Relevant results for the work presented are: the optimality of base-stock policies for simple cases proved in [ClarkScarf60], [FergZip86a] [FergZip86b], and [RodKap04]; the practical approach used in [GlasTay94] to determine stability conditions, and in [GlasTay95] to validate sensitivity analysis for a single product multi-echelon capacitated production system using a base-stock policy; the study of several common base-stock policies applied to re-entrant multi-product capacitated systems in [Bispo97]; the first practical study of smooth priorities in [NunSouSou99]; and the optimality of base-stock policies in the multiple products, single machine case approached in [JanNagVeer07].

To obtain practical results, a simulation based optimization with gradient estimates provided by Infinitesimal Perturbation Analysis, (IPA), ([Glass91]) is used. IPA is based on the possibility of exchanging the expectation and the derivative operators allowing for a gradient to be estimated by the average of the period gradients calculated during simulation. The optimizer was developed after a review of several algorithms from [Luen03] and uses the Fletcher-Reeves method to determine the search direction. Constraints are enforced by projection and the golden section algorithm is used for line searches. The simulator was developed by adaptation of the algorithm and techniques from [CasLaf99]. More information on the implementation details is available in [Men08].

In what follows we will use simple substitution to prove that our implementation of smooth priorities is a generalization of both the Linear Scaling Rule, [Bispo97], (LSR) and strict priorities.

The general model used along with the LSR, strict priorities, and smooth priorities production equations, and relevant IPA equations are presented in Section 2. The fact that LSR and strict priorities for two products constitute a subclass of smooth priorities is proven in Section 3. Practical cases studied, results obtained, and drawn conclusions are presented in Section 4. Conclusions are presented in Section 5.

2. MODEL AND BASE-STOCK POLICIES

The general model used in this paper has M machines with finite capacity, P final products that are constituted by F_p phase products. Each phase product can be assigned to be produced in any machine without restriction (i.e., no serial or other structure is imposed) and can use up different amounts of capacity to be produced (i.e., different loads), raw materials are always available (i.e., perfect delivery), which are used by the first phase product of a final product, demand is continuous and occurs on the last phase product of a final product after production. Product quantities are considered continuous and linear holding and backlog costs are used.

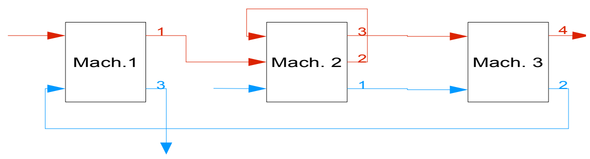

Figure 1 - Illustration of the modeling flexibility.

Notation:

$\mathcal{M}$ - number of machines;

$\mathcal{P}$ - number of final products;

$\mathcal{F}_p$ - number of phase products of final product p ;

M_m - set of phase products produce in machine m ;

K_m - capacity of machine m ;

$I_t^{p,f}$ - inventory of phase product f of final product p in time period t ;

$E_t^{p,f}$ - echelon inventory of phase product f of final product p in time period t ;

$P_t^{p,f}$ - production of phase product f of final product p in time period t ;

d_t^p - demand for final product f at the end of time period t ;

$h^{p,f}$ - holding cost of phase product f of final product p ;

b^p - backlog cost of final product p ;

$\tau^{p,f}$ - load of phase product f of final product p ;

C_t^p - cost for final product p in time period t ;

C_t - total cost in time period t ;

$z^{p,f}$ - echelon base-stock level of phase product f of final product p ;

$\Delta^{p,f}$ - alternative (to $z^{p,f}$) set of variables that relate inventory between phases;

$y_t^{p,f}$ - shortfall of phase product f of final product p in time period t ;

$f_t^{p,f}$ - production needs of phase product f of final product p in time period t ;

$\alpha^{p,f}$ - smooth priorities' α parameter of phase product f of final product p .

Equations that show how the system evolves from period to period:

$$I_{t+1}^{p,f} = I_t^{p,f} + P_t^{p,f} - P_t^{p,f+1} \quad , P^{p,\mathcal{F}_p+1} \equiv d_t^p \tag{1}$$

$$E_{t+1}^{p,f} = E_t^{p,f} + P_t^{p,f} - d_t^p \quad , E_t^{p,f} = \sum_{i=f}^{\mathcal{F}_p} (I_t^{p,i}) \tag{2}$$

Equations to determine the cost in each period:

$$C_t^p \; = \; -min(I_t^{p,\mathcal{F}_p},0) \times b^p + \sum_{f=1}^{\mathcal{F}_p} (max(I_t^{p,f},0) \times h^{p,f}) \tag{3}$$

$$C_t = \sum_{p=1}^{\mathcal{P}} (C_t^p) \tag{4}$$

Equations that define $\Delta^{p,f}$, $y_t^{p,f}$ and $f_t^{p,f}$:

$$\Delta^{p,f} = z^{p,f} - z^{p,f+1} \quad , \quad z^{p,\mathcal{F}_p+1} \equiv 0 \tag{5}$$

$$y_t^{p,f} = max(z^{p,f} - E_t^{p,f},0) \tag{6}$$

$$f_t^{p,f} = min(y_t^{p,f}, I_t^{p,f-1}) \tag{7}$$

2.1 Description of base-stock policies and their production equations

Linear Scaling Rule (LSR) - when production needs cannot be satisfied for all products of a given machine, production is linearly scaled to fit the machine capacity.

$$P_t^{p,f} = \begin{cases} f_t^{p,f}, & \sum_{(i,j)\in M_m} f_t^{i,j} \tau^{i,j} \leq K_m \\ f_t^{p,f} \dfrac{K_m}{\sum_{(i,j)\in M_m} f_t^{i,j} \tau^{i,j}}, & \sum_{(i,j)\in M_m} f_t^{i,j} \tau^{i,j} > K_m \end{cases} \tag{8}$$

Strict Priority - each product has a priority assigned and fulfills production necessities by order of priority until machine capacity is fully used or there are no more products to produce. In the following equation we consider products ordered by priority and j=1 means the product has the highest priority.

$$P_t^j = min(f_t^j, \frac{K_m - \sum_{i=0}^{j-1} P^i \tau^i}{\tau^j}) \quad , \quad P^0 \equiv 0 \tag{9}$$

Whereas we have addressed the drawbacks of strict priorities in the Introduction, we need now to refer that the LSR may perform well under certain circumstances, see [Bispo97], but when first phase products enter the scaling term of (8), they tend to crush the other products just because there is no bound in raw materials. That is, while their net needs, $f_t^{p,f}$, are exactly equal to the shortfall, the net needs of other product may be bounded by the feeding inventory. This is specially relevant for re-entrant systems. Therefore, the LSR ends up working as if giving higher priority to the first phase products, which is known not to be a good production strategy.

The idea behind the smooth priorities concept is that at any decision point all products should have a higher chance of getting part of their needs satisfied. When deciding with strict priorities the smaller priorities products quite often do not get any of their needs satisfied. This happens when the higher priority products exhaust all available capacity. Therefore, to circumvent this drawback, we propose that the production needs for all products should be considered in the decision, proportionally to their relative priorities. This way we reduce the likelihood of the lower priority products not getting anything done at any given period.

To introduce an implementation of the smooth priorities concept, we resort to the LSR. There, each product needs enter the scaling term of (8) with equal weight. This means all products needs are equally important. If instead we assign a weight, between 0 and 1, to each product we may be able to regulate the relative importance of each product in the scaling term. With this idea in mind we introduce now an implementation of the smooth priorities concept.

Smooth Priorities – Each production decision at any given period has two decision steps where each is produced with LSR. Each product has a parameter α that indicates the relative quantity of production needs that enter the first phase LSR. If there is remaining capacity after the first phase, $(1-\alpha)$ is the proportion that enters the second phase LSR.

$$P_{t,I}^{p,f} = \alpha^{p,f} f_t^{p,f} min(\frac{K_m}{\sum_{(i,j)\in M_m} \alpha^{i,j} f_t^{i,j} \tau^{i,j}}, 1) \tag{10}$$

$$P_{t,II}^{p,f} = (1-\alpha^{p,f}) f_t^{p,f} min(\frac{K_m - \sum_{(i,j)\in M_m} P_{t,I}^{i,j} \tau^{i,j}}{\sum_{(i,j)\in M_m} (1-\alpha^{i,j}) f_t^{i,j} \tau^{i,j}}, 1) \tag{11}$$

$$P_t^{p,f} = P_{t,I}^{p,f} + P_{t,II}^{p,f} \tag{12}$$

It should be easy to see now that, in any period when the sum of the net needs for all products at a given machine exceed its capacity, if all the relative weights are different from zero, all products will get a non zero share of the available capacity, contrary to what happens with the strict priorities policy. Also, given the fact that in general it is not a good strategy to give higher priority to first phase products, one should expect that the optimal weights should reflect this fact by reducing adequately their values for first phase products.

2.2 IPA equations

The IPA equations are omitted here due to space constraints, but can be seen in [Men08]. They are easily obtained applying the formal procedure described in [GlasTay95]. IPA equations and their validation for several policies in the multi-product case is done in [Bispo97]. Since, our implementation of the smooth priorities is essentially an LSR scheme in two phases, the validation procedure for that policy carries through for the present case.

3. THEORETICAL COMPARISON OF POLICIES

To show that the LSR policy is contained in the smooth priorities policy we show the equality of both when smooth priorities has the α parameters equal for all products.

The smooth priorities production equations (10), (11), and (12) can be expressed differently as a disjunction of all the possible cases which gets rid of the min operators.

$$\overset{Smooth}{P}{}_t^{p,f} = \begin{cases} f_t^{p,f} & if \sum_{(i,j)\in M_m} f_t^{i,j}\tau^{i,j} \le K_m \\ \alpha^{p,f} f_t^{p,f} + (1-\alpha^{p,f}) f_t^{p,f} \frac{K_m - \sum_{(i,j)\in M_m} \alpha^{i,j} f_t^{i,j}\tau^{i,j}}{\sum_{(i,j)\in M_m} (1-\alpha^{i,j}) f_t^{i,j}\tau^{i,j}} & \\ \quad if \sum_{(i,j)\in M_m} f_t^{i,j}\tau^{i,j} > K_m \wedge \sum_{(i,j)\in M_m} \alpha^{i,j} f_t^{i,j}\tau^{i,j} \le K_m & \\ \alpha^{p,f} f_t^{p,f} \frac{K_m}{\sum_{(i,j)\in M_m} \alpha^{i,j} f_t^{i,j}\tau^{i,j}}, & \\ \quad if \sum_{(i,j)\in M_m} f_t^{i,j}\tau^{i,j} > K_m \wedge \sum_{(i,j)\in M_m} \alpha^{i,j} f_t^{i,j}\tau^{i,j} > K_m & \end{cases} \tag{13}$$

By doing $\forall (p,f)\in M_n \quad \alpha^{p,f} = \alpha$, simplifying and joining the equally valued disjunction we finally arrive at (14).

$$\overset{Smooth}{P}{}_t^{p,f} = \overset{LSR}{P}{}_t^{p,f} = \begin{cases} f_t^{p,f}, & \sum_{(i,j)\in M_m} f_t^{i,j}\tau^{i,j} \le K_m \\ f_t^{p,f} \frac{K_m}{\sum_{(i,j)\in M_m} f_t^{i,j}\tau^{i,j}}, & \sum_{(i,j)\in M_m} f_t^{i,j}\tau^{i,j} > K_m \end{cases} . \tag{14}$$

To show that for two products strict priorities are contained in smooth priorities we show the equality of both when the highest strict priority product has $\alpha = 1$ and the lowest strict priority product has $\alpha = 0$.

For product 1 with $\alpha = 1$ we get a null second phase of production and for product 2 with $\alpha = 0$ we get a null first phase giving the following equations

$$P^{Smooth}{}_t^1 = f_t^1 \, min(\frac{K_m}{f_t^1 \tau^1}, 1) \quad (15)$$

$$P^{Smooth}{}_t^2 = f_t^2 \, min(\frac{K_m - P_{t,I}^1 \tau^1}{f_t^2 \tau^2}, 1) \quad (16)$$

The equations for the strict priority case with only two products are the following

$$P^{Strict}{}_t^1 = min(f_t^1, \frac{K^m}{\tau^1}) \quad (17)$$

$$P^{Strict}{}_t^2 = min(f_t^2, \frac{K_m - P^1 \tau^1}{\tau^2}) \quad (18)$$

It can easily be seen that they are equivalent. It is also important to note that by extending smooth priorities to have as many LSR phases as there are products in the machine this proof could easily be generalized for any amount of products.

4. NUMERICAL RESULTS

First, the previous results of [NunSouSou99] are reproduced as closely as possible. This is done with the purpose of confirming the validity of previous results and validating the approach here proposed. Then a new simple case is studied for several costs. This serves the purpose of showing that strict priorities constitute upper bounds in terms of cost relative to the smooth priorities, as was established in Section 3, while showing that there is a choice of smoothing parameters that equals the behaviour of strict priorities for only two products. A more complex case is studied and a parallel to the results of [LuKu91] is made. To finish, the structure of the cost function is studied and the extrapolation for the general case is attempted.

4.1 Comparison with Previous Results

The case is this:
- one machine, with capacity 25, produces two products
- product one has: average demand = 8; inverse variance coefficient = 3; backlog cost = 50; and holding cost = 10.
- product two has: average demand = 12; inverse variance coefficient = 1; backlog cost = 20; and holding cost = 10.

This is a case in which, according to [Bispo97], a strict priority policy would have better results than the LSR and another rule proposed there but omitted here, the ESR (Equalized Shortfall Rule), since one of the products has the lowest average demand, the highest backlog costs, and the least variance.

The results obtained by the optimizer for strict priority for 1st product, smooth priority and strict priority for 2nd product are presented in Table 1. The first and third line of the table refer to the results obtained for strict priorities and the middle line refers to smooth priorities. In the former case we only optimize the base-stocks for both products, and in the latter we optimize the base-stocks and the weighting parameters. The behaviour is as expected in terms of the relative priority but the best performance is not obtained with strict priorities since we do not have $\alpha^1 = 1$ nor $\alpha^2 = 0$ for the smooth priorities. Actually, the fact that the optimal weighting factor for product 2 is non zero, means that this product will have access to capacity in every period when its needs are non zero, contrary to what would happen under strict priorities.

Note also that, as expect from the discussion made in the Introduction, the performance improvement is done at the expense of increasing slightly the base-stock for product 1 and reducing it for product 2, when compared with the first line. That is, product 2 does not have to hedge as much given it has a more reliable access to the production resource.

Table 1. Comparison between strict and smooth priorities

Optimization Results				
Δ^1	Δ^2	α^1	α^2	Avg. Cost
10.513	25.6131	1	0	299.484
11.9863	23.764	0.54975	0.01064	292.810
26.2372	16.1068	0	1	444.731

4.2 Different Costs

The case is this:
- one machine, with capacity 100, produces two products
- product one has: average demand = 40; and exponential demand distribution.
- product two has: average demand = 40; and exponential demand distribution.
- the holding and backlog costs are varied.

We present a summary of the findings. Results show that, for the cases where holding and backlog costs are similar for both products, smooth priorities clearly achieves better results and is equivalent to the use of the LSR. This was to be expected since there is no differentiating factor between the products.

In general, smooth priorities achieves better results than strict priorities but do converge to strict priorities in several of the cases where holding costs differ. For every case the ordering of the strict priorities with the best results correspond to ordering of the alphas for smooth priorities. As would be expected, when holding or backlog costs rise for a product, smooth priorities get closer to strict priorities for that product.

This practically confirms that the performance of strict priorities constitutes an upper bound on the performance of smooth priorities as shown in Section 3. Given the extent of the data, for the sake of space, we omit presenting the table with all the results here. These can be seen in [Men08].

4.3 Complex System

To explore the use of smooth priorities in a more complex structure, the system illustrated in Figure 1 is used.

The details are the following:

- 3 machines
- machine capacities: $K_1 = 60$, $K_2 = 100$, $K_3 = 65$
- 1st final product: average demand = 30; exponential demand distribution; 4 phase products with holding costs: $h^{1,1} = 10$, $h^{1,2} = 15$, $h^{1,3} = 20$, $h^{1,4} = 25$; and backlog cost = 50.
- 2nd final product: average demand = 20; exponential demand distribution; 3 phase products with holding costs: $h^{2,1} = 10$, $h^{2,2} = 10$, $h^{2,3} = 10$; and backlog cost = 20.
- the first machine produces: phase product 1 of final product 1; and phase product 3 of final product 2.
- the second machine produces: phase product 2 of final product 1; phase product 3 of final product 1; and phase product 1 of final product 2.
- the third machine produces: phase product 4 of final product 1; and phase product 2 of final product 2.

The system was optimized with all the machines with smooth priorities and then with machine one with strict priority for product 2 and the remaining machines with smooth priorities. The results obtained (rounded) are shown in Table 2.

Table 2. Smooth vs strict priority (product 2, machine 1)

Optimization Results														
Δ							α							Cost
prod.1				prod.2			prod.1				prod.2			
f1	f2	f3	f4	f1	f2	f3	f1	f2	f3	f4	f1	f2	f3	
41	39	37	89	4	44	53	.45	.59	1.0	.99	.20	.00	.99	4855
57	16	47	82	7	37	49	.00	.68	1.0	.92	.21	.30	1.0	4927

From the results we can see that the alpha parameters give priority to the phase products closer to the external demand. This brings to mind the "Last Buffer First Served" strict priorities policy that in [LuKu91] is the one that achieves the best performance (in terms of mean cycle time), from those studied there, in a system with a re-entrant structure.

We should also take notice that applying strict priorities in machine 1 when the alpha parameters of smooth priorities give clear order of priority, but not strict priority, leads to an increase in the cost.

4.4 Structure of the cost function

Several graphs where produced, for both one of the simple and the more complex systems, to empirically study the structure of the cost function. Given the number of parameters involved, we can only look at projections of the cost function. Therefore, we have to select a pair of parameters to produce a cost surface as a function of the pair, while the remaining parameters are kept constant.

We show those that present the most interesting behaviour for the single machine and two products in Fig. 2 and for the system of Fig. 1 in Fig. 3.

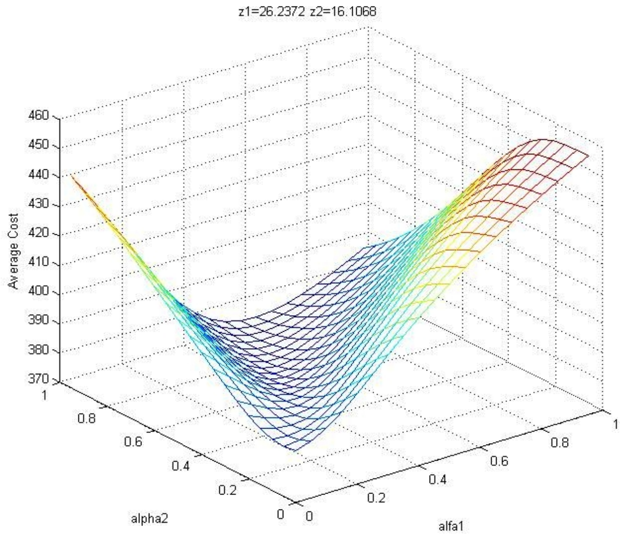

Figure 2 - Cost function for varying alphas (single machine).

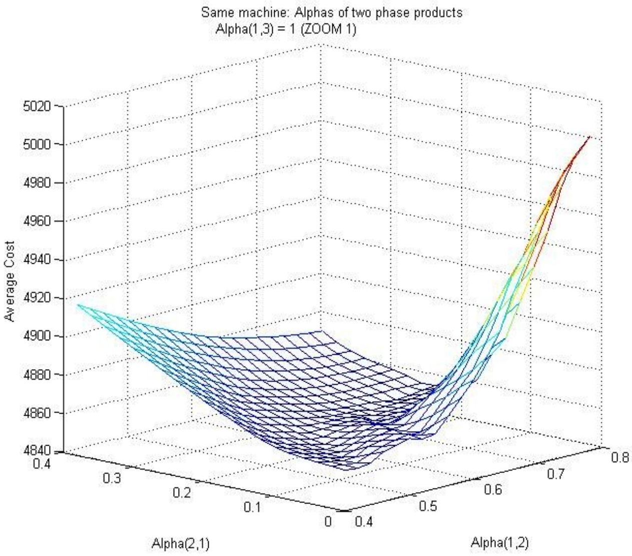

Figure 3 - Cost function for varying alphas (Same machine for the system of Fig.1).

Besides the two here displayed, all graphs produced under the above mentioned settings present smooth surfaces. Although this does not constitute proof, these observations together with the conditions which allow the validation of the IPA approach reinforce the belief that the same happens in general. Although the graphs for a single machine do not present local minima, which permits us to speculate that the function is quasi-convex for that case, the graph in Fig. 3 exhibits local minima. Since from all the graphs we have produced this is the only one for a machine with more than two phase products but also in a re-entrant system, this raises the question if such behaviour is due to the re-entrance or if it does happen even for three products with no re-entrance.

When we started this line of research one of the goals was to investigate if the cost function has a single minimum in general. We selected the above pair of graphs to show two things: as a function of the alpha parameters, the cost is not convex and may present local minima.

The numerical results are therefore coherent with the theoretical expectations that the cost function is continuous but not with the expectations that it is quasi-convex, at least generally. This, unfortunately, means that gradient based optimization is not suitable to obtain the optimal parameters for sufficiently general systems. Therefore, other optimization techniques will have to be used in order to satisfactorily ensure optimality of the parameters obtained. This fact also bears consequences into the purpose of using smooth priorities to obtain the optimal set of priorities if one still intends to resort to strict priorities. Although global optimality cannot be ensured with gradient based optimization, we still believe that the error committed in the precise value of the alpha parameters, under gradient based optimization, does not carry through to their relative values. Thus, for that purpose we continue to claim that a combinatorial problem has been converted into a non linear problem.

5. CONCLUSIONS

A general system model that can encompass both serial and re-entrant systems was presented. Software was developed that implements the functioning of the systems, a general simulator and a general optimizer allowing for their optimization based on simulation.

Equations that express the functioning of the system were presented and were used to theoretically show that smooth priorities are a generalization of the Linear Scaling Rule and of the strict priorities for systems where each machine processes only two different products.

With the use of the developed software, previous numerical results where confirmed. The fact that smooth priorities consistently achieve better results than strict priorities was verified. The hypothesis that the ordering of the alphas under smooth priorities translate to an optimal ordering of the products for strict priorities was reinforced and the structure of the cost function was studied, leading to the conclusion that it is most likely continuous but, unfortunately, it is not quasi-convex in general.

Future work will have to address other implementations of the smooth priorities concept that may be more amenable to optimization, given the fact that the numerical results show beyond doubt that there are gains to expect when departing from strict priorities.

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