A nonlinear position and attitude observer on SE(3) using landmark measurements

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Abstract

This paper addresses the problem of position and attitude estimation, based on landmark readings and velocity measurements. A derivation of a nonlinear observer on SE(3) is presented, using a Lyapunov function conveniently expressed as a function of the difference between the estimated and the measured landmark coordinates. The resulting feedback laws are explicit functions of the landmark measurements and velocity readings, exploiting the sensor information directly in the observer. The proposed observer yields almost global asymptotic stabilization of the position and attitude errors and exponential convergence in any closed ball inside the region of attraction. Also, it is shown that the asymptotic convergence of the estimation error trajectories is shaped by the landmark geometry and observer design parameters. The problem of non-ideal velocity readings is also considered, and the observer is augmented to compensate for bias in the angular and linear velocity measurements. The resulting position, attitude, and bias estimation errors are shown to converge exponentially fast to the desired equilibrium points, for bounded initial estimation errors. Simulation results are presented to illustrate the stability and convergence properties of the observer.

Key words:

Nonlinear observer, Lyapunov stability, Bias filtering, Exponential convergence, Linear time-varying systems

1. Introduction

Landmark based navigation is recognized as a promising strategy for providing autonomous vehicles with accurate position and attitude information during critical operation stages such as take-off, landing, or docking. Among a wide diversity of suitable estimation techniques, nonlinear observers stand out as an exciting approach often endowed with stability results [1], and formulated rigorously in non-Euclidean spaces. Research on the problem of deriving a stabilizing law for systems evolving on manifolds, such as SO(3) and SE(3), can be found in [2, 3, 4, 5, 6, 7, 8], where the topological limitations for achieving global stabilization on SO(3) provide important guidelines for the design of observers. In particular, these limitations call for the relaxation from global to almost global stability, as adopted in [9, 10], meaning that the region of attraction of the origin comprises all the state space except a nowhere dense set of measure zero [11].

Nonlinear attitude and position observers, with application to aerospace, terrestrial and oceanic vehicles, have been proposed in recent literature. A seminal work on nonlinear attitude observers can be found in [12], where the author proposes a solution to estimate attitude in the Euler quaternion representation, using attitude and torque measurements. Subsequent work has been devoted to nonlinear observers based on the attitude and position kinematics [13, 14, 15, 16, 17], which can be implemented on any robotic platform, irrespective of its dynamics. Some methodologies for the design of kinematic observers evolving on manifolds have been put forth in [7, 11, 18, 19].

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Another line of work has been directed at the design of nonlinear observers that exploit information sources adopted in classical navigation problems [15, 16, 17, 20], and compensate for non-ideal sensor readings that have long been considered in filtering estimation techniques.

This work explores the circle of ideas found in [19] and [21], that were proposed to address the problem of rigid-body stabilization based on landmark measurements and ideal velocity readings. In this paper, the problem of attitude and position estimation on SE(3) is addressed and a nonlinear observer based on landmark measurements and possibly biased velocity readings is proposed. The observer is derived constructively using a conveniently defined Lyapunov function, defined by the landmark estimation error. For ideal velocity readings, the obtained feedback law yields almost global asymptotic stability of the desired equilibrium point on SE(3), and exponential convergence of the attitude and position estimates in any closed ball inside the region of attraction.

The adopted derivation technique yields feedback laws that are an explicit function of the sensor measurements. The direct use of sensor readings in the feedback law provides for a geometric insight on the observer properties. Namely, it shows that the asymptotic behavior of the estimation errors can be shaped by judicious landmark placement and design parameter tuning. Also, it highlights the necessary landmark configuration for attitude and position estimation.

The problem of bias in the velocity readings is addressed by extending the proposed observer to dynamically compensate for these sensor non-idealities. Exponential stabilization of the position and attitude estimation errors is obtained, for worst-case initial estimation errors. Recent results for parameterized linear time-varying systems [22] are adopted in the stability analysis of the system. Simulation results validate the proposed observer, and illustrate the derived properties. A preliminary version of this work has been presented in [23].

The paper is organized as follows. In Section 2, the position and attitude estimation problem is introduced and the available sensor information is detailed. The attitude and position observer is derived in Section 3. A convenient landmark-based Lyapunov function is defined, and the necessary and sufficient landmark configuration for attitude estimation is discussed. Almost global stabilizing feedback laws are obtained for attitude and position estimation. The resulting observer dynamics are expressed as a function of the sensor readings, and it is shown that the asymptotic convergence of the system trajectories is determined by the landmark geometry, and by the design parameters. The problem of unknown velocity sensor bias is studied in Section 4. The observer dynamics are extended to dynamically compensate for the bias in the linear and angular velocity measurements, and stability results are derived. In Section 5, simulation results illustrate the observer properties for time-varying linear and angular velocities. Concluding remarks are presented in Section 6.

Nomenclature

The notation adopted is fairly standard. Column vectors and matrices are denoted respectively by lowercase and uppercase boldface type, e.g. $\mathbf{s}$ and $\mathbf{S}$. The transpose of a vector or matrix will be indicated by a prime, e.g. $\mathbf{s}'$ and $\mathbf{S}'$. The set of $n \times m$ matrices with real entries is denoted by $\mathrm{M}(n,m)$ and $\mathrm{M}(n) := \mathrm{M}(n,n)$. The sets of orthogonal, special orthogonal, symmetric, and skew-symmetric matrices are denoted by $\mathrm{O}(n) := \{\mathbf{U} \in \mathrm{M}(n) : \mathbf{U}'\mathbf{U} = \mathbf{I}\}$, $\mathrm{SO}(n) := \{\mathbf{R} \in \mathrm{O}(n) : \det(\mathbf{R}) = 1\}$, $\mathrm{L}(n) := \{\mathbf{S} \in \mathrm{M}(n) : \mathbf{S} = \mathbf{S}'\}$, and $\mathrm{so}(n) := \{\mathbf{S} \in \mathrm{M}(n) : \mathbf{S} = -\mathbf{S}'\}$, respectively. The special Euclidean group is given by the product space of $\mathrm{SO}(n)$ with $\mathbb{R}^n$, $\mathrm{SE}(n) := \mathrm{SO}(n) \times \mathbb{R}^n$ [24]. The n-dimensional unit sphere and ball are described by $\mathrm{S}(n) := \{\mathbf{x} \in \mathbb{R}^{n+1} : \mathbf{x}'\mathbf{x} = 1\}$ and $\mathrm{B}(n) := \{\mathbf{x} \in \mathbb{R}^n : \mathbf{x}'\mathbf{x} \leq 1\}$, respectively. The operator $(\mathbf{a})_\times : \mathbb{R}^3 \to \mathrm{so}(3)$ yields the skew symmetric matrix defined by the vector $\mathbf{a} \in \mathbb{R}^3$ such that $(\mathbf{a})_\times \mathbf{b} = \mathbf{a} \times \mathbf{b}$, $\mathbf{b} \in \mathbb{R}^3$. The inverse of $(\cdot)_\times$ is denoted by $(\cdot)_\otimes$, i.e. $((\mathbf{a})_\times)_\otimes = \mathbf{a}$.

2. Problem formulation

Landmark based navigation, illustrated in Fig. 1, can be summarized as the problem of determining attitude and position of a

Figure 1: Landmark based navigation.

rigid body using landmark observations and velocity measurements, given by sensors installed onboard the robotic platform. The rigid body kinematics are described by

$$\dot{\mathcal{R}} = \mathcal{R}\,(\omega)_\times\,, \quad \dot{\mathbf{p}} = \mathbf{v} - (\omega)_\times\,\mathbf{p}, \tag{1}$$

where $\mathcal{R}$ is the shorthand notation for the rotation matrix ${}^L_B\mathbf{R}$ from body frame $\{B\}$ to local frame $\{L\}$ coordinates, $\boldsymbol{\omega}$ and $\mathbf{v}$ are the shorthands for ${}^B\boldsymbol{\omega}$ and ${}^B\mathbf{v}$, the body angular and linear velocities, respectively, expressed in $\{B\}$, $\mathbf{p}$ is the shorthand for ${}^B\mathbf{p}$, the position of the rigid body with respect to $\{L\}$ expressed in $\{B\}$, and $\dot{\mathbf{p}}$ denotes the time derivative of $\mathbf{p}$, that is $\frac{\mathrm{d}\,{}^B\mathbf{p}}{\mathrm{dt}}$.

The body angular and linear velocities are measured by a rate gyro sensor triad and a Doppler sensor, respectively

$$\omega_r = \boldsymbol{\omega} + \mathbf{b}_\omega, \quad \mathbf{v}_r = \mathbf{v} + \mathbf{b}_v, \tag{2}$$

where the nominal biases are unknown and constant, i.e. $\dot{\mathbf{b}}_\omega = \mathbf{0}$, $\dot{\mathbf{b}}_v = \mathbf{0}$. The landmark measurements, denoted as $\mathbf{q}_i$ and illustrated in Fig. 1, are obtained by on-board sensors that are able to track terrain characteristics, such as CCD cameras or ladars,

$$\mathbf{q}_i = \mathcal{R}'\,{}^L\mathbf{x}_i - \mathbf{p}, \tag{3}$$

where ${}^L\mathbf{x}_i$ represent the coordinates of landmark i in the local frame $\{L\}$. The concatenation of (3) is expressed in matrix form as $\mathbf{Q} = \mathcal{R}'\mathbf{X} - \mathbf{p}\mathbf{1}_n'$, where $\mathbf{Q} = [\mathbf{q}_1 \;\dots\; \mathbf{q}_n]$, $\mathbf{X} = \left[{}^L\mathbf{x}_1 \;\dots\; {}^L\mathbf{x}_n\right]$, $\mathbf{Q}, \mathbf{X} \in \mathrm{M}(3, n)$.

Without loss of generality, the origin of the local frame is defined at the landmarks centroid, as depicted in Fig. 1, bearing

$$\sum_{i=1}^{n} {}^L\mathbf{x}_i = \mathbf{X}\mathbf{1}_n = 0. \tag{4}$$

The proposed observer reproduces the rigid body kinematics (1), taking the form

$$\dot{\hat{\mathcal{R}}} = \hat{\mathcal{R}}\,(\hat{\omega})_\times\,, \quad \dot{\hat{\mathbf{p}}} = \hat{\mathbf{v}} - (\hat{\omega})_\times\,\hat{\mathbf{p}}, \tag{5}$$

where $\hat{\omega}$ and $\hat{\mathbf{v}}$ are the feedback terms constructed to compensate for the attitude and position estimation errors.

The position and attitude errors are defined as $\tilde{\mathbf{p}} := \hat{\mathbf{p}} - \mathbf{p}$ and $\tilde{\mathcal{R}} := \hat{\mathcal{R}}\mathcal{R}'$, respectively. The Euler angle-axis parametrization of the rotation error matrix $\tilde{\mathcal{R}}$ is described by the rotation vector $\boldsymbol{\phi} \in \mathrm{S}(2)$ and by the rotation angle $\varphi \in [0\ \pi]$, yielding the DCM formulation [24], denoted by $\tilde{\mathcal{R}} = \mathrm{rot}(\varphi, \boldsymbol{\phi})$ and given by

$$\mathrm{rot}(\varphi, \boldsymbol{\phi}) = \cos(\varphi)\mathbf{I} + \sin(\varphi)\,(\boldsymbol{\phi})_\times + (1 - \cos(\varphi))\boldsymbol{\phi}\boldsymbol{\phi}'. \tag{6}$$

In this work, the observer is designed and analyzed on the SE(3) manifold. The Euler angle-axis parametrization is used solely to characterize interesting properties of the estimation error trajectories.

The attitude and position error dynamics are a function of the linear and angular velocity estimates and given by

$$\dot{\tilde{\mathcal{R}}} = \tilde{\mathcal{R}}\,(\mathcal{R}(\hat{\omega} - \omega))_\times\,, \tag{7a}$$

$$\dot{\tilde{\mathbf{p}}} = (\hat{\mathbf{v}} - \mathbf{v}) - (\omega)_\times\,\tilde{\mathbf{p}} + (\hat{\mathbf{p}})_\times\,(\hat{\omega} - \omega). \tag{7b}$$

The attitude and position feedback laws are obtained by defining $\hat{\omega}$ and $\hat{\mathbf{v}}$ as a function of the velocity readings (2) and landmark observations (3), so that the closed loop position and attitude estimation errors converge asymptotically to the origin, i.e. $\tilde{\mathcal{R}} \to \mathbf{I}, \tilde{\mathbf{p}} \to 0$ as $t \to \infty$.

The attitude and position feedback laws are derived first for the case of unbiased velocity measurements ($\mathbf{b}_\omega = \mathbf{b}_v = \mathbf{0}$), and the case of biased linear and angular velocity measurements ($\mathbf{b}_\omega, \mathbf{b}_v$ unknown) is subsequently considered.

3. Observer synthesis with ideal velocity measurements

In this section, the attitude and position feedback laws are derived for the case of ideal angular and linear velocity measurements, where $\mathbf{b}_\omega = \mathbf{b}_v = \mathbf{0}$. The closed loop system is demonstrated to have an almost GAS equilibrium point at $(\tilde{\mathcal{R}}, \tilde{\mathbf{p}}) = (\mathbf{I}, 0)$, which is exponentially stable in any closed ball inside the region of attraction.

Some relevant characteristics of the observer are pointed out. It is shown that the position and attitude feedback laws can be expressed as an explicit function of the sensor readings, allowing for the observer implementation in practice. Also, the asymptotic behavior of the attitude observer trajectories is studied in the Euler angle-axis representation, characterizing the directionality of the observer estimates given the design feedback law.

3.1. Landmark based Lyapunov function

The observer is derived resorting to Lyapunov's stability theory. To exploit the landmark readings information, vector and position measurements are constructed from a linear combination of (3), producing respectively

$$ {}^{B}\mathbf{u}_j = \sum_{i=1}^{n-1} a_{ij}(\mathbf{q}_{i+1} - \mathbf{q}_i), \quad {}^{B}\mathbf{u}_n = -\frac{1}{n}\sum_{i=1}^{n}\mathbf{q}_i, \tag{8} $$

where $j = 1, \dots, n-1$. The transformation (8) can be expressed in matrix form as

$$ {}^{B}\mathbf{U} = \mathbf{QDA}, \quad {}^{B}\mathbf{u}_n = \mathbf{Qd}_p, \tag{9} $$

where ${}^{B}\mathbf{U} = \begin{bmatrix} {}^{B}\mathbf{u}_1 & \dots & {}^{B}\mathbf{u}_{n-1} \end{bmatrix} \in \mathrm{M}(3, n-1)$, $\mathbf{D} = \begin{bmatrix} \mathbf{0}_{1\times n-1} \\ \mathbf{I}_{n-1} \end{bmatrix} - \begin{bmatrix} \mathbf{I}_{n-1} \\ \mathbf{0}_{1\times n-1} \end{bmatrix}$, $\mathbf{D} \in \mathrm{M}(n, n-1)$, $\mathbf{d}_p = -\frac{1}{n}\mathbf{1}_n$, and the linear transformation $\mathbf{A} = [a_{ij}] \in \mathrm{M}(n-1)$ is considered invertible by construction.

Using the fact that $\mathbf{1}_n'\mathbf{D} = \mathbf{0}$, and that (4) implies $\mathbf{Xd}_p = \mathbf{0}$, (9) can be rewritten as

$$ {}^{B}\mathbf{U} = \mathcal{R}'\mathbf{U}, \quad {}^{B}\mathbf{u}_n = \mathbf{p}, $$

where $\mathbf{U} = \mathbf{XDA}$. Consequently, ${}^{B}\mathbf{U}$ and ${}^{B}\mathbf{u}_n$ can be reconstructed using the observer estimates as follows

$$ {}^{B}\hat{\mathbf{U}} = \hat{\mathcal{R}}'\mathbf{U}, \quad {}^{B}\hat{\mathbf{u}}_n = \hat{\mathbf{p}}, \tag{10} $$

which can be partitioned in columns ${}^{B}\hat{\mathbf{U}} = \begin{bmatrix} {}^{B}\hat{\mathbf{u}}_1 & \dots & {}^{B}\hat{\mathbf{u}}_{n-1} \end{bmatrix}$, and $\mathbf{U} = \begin{bmatrix} {}^{L}\mathbf{u}_1 & \dots & {}^{L}\mathbf{u}_{n-1} \end{bmatrix}$.

The candidate Lyapunov function is defined by the estimation error of the transformed vectors

$$ V = \frac{1}{2}\sum_{i=1}^{n} \|{}^{B}\hat{\mathbf{u}}_i - {}^{B}\mathbf{u}_i\|^2, \tag{11} $$

that can be described as a sum of distinct position and attitude components.

Proposition 1. *The Lyapunov function* (11) *can be written as* $V = V_{\mathcal{R}} + V_p$*, where the distinct attitude and position components are respectively given by*

$$ V_{\mathcal{R}} = \mathrm{tr}\left[(\mathbf{I} - \tilde{\mathcal{R}})\mathbf{UU}'\right] = \frac{1}{4}\|\mathbf{I} - \tilde{\mathcal{R}}\|^2 \boldsymbol{\phi}'\mathbf{P}\boldsymbol{\phi}, \tag{12a} $$

$$ V_p = \frac{1}{2}\tilde{\mathbf{p}}'\tilde{\mathbf{p}}. \tag{12b} $$

and $\mathbf{P} = \mathrm{tr}(\mathbf{UU}')\mathbf{I} - \mathbf{UU}'$.

Proof. The decoupling is obtained by defining the attitude and position components as

$$ V_{\mathcal{R}} = \frac{1}{2}\sum_{i=1}^{n-1}\|{}^{B}\hat{\mathbf{u}}_i - {}^{B}\mathbf{u}_i\|^2, \quad V_p = \frac{1}{2}\|{}^{B}\hat{\mathbf{u}}_n - {}^{B}\mathbf{u}_n\|^2. \tag{13} $$

producing $V = V_{\mathcal{R}} + V_p$ immediately from (11).

The attitude component is expressed as a function of the attitude error by rewriting $V_{\mathcal{R}}$ as

$$ \begin{aligned} V_{\mathcal{R}} &= \frac{1}{2}\|{}^{B}\hat{\mathbf{U}} - {}^{B}\mathbf{U}\|^2 = \frac{1}{2}\|(\hat{\mathcal{R}}' - \mathcal{R}')\mathbf{U}\|^2 \\ &= \frac{1}{2}\|(\mathbf{I} - \tilde{\mathcal{R}})\mathbf{U}\|^2 = \frac{1}{2}\,\mathrm{tr}\left[(\mathbf{I} - \tilde{\mathcal{R}})'\mathbf{UU}'(\mathbf{I} - \tilde{\mathcal{R}})\right]. \end{aligned} $$

Using the trace properties presented in Appendix A, and the DCM expansion (6) yields

$$ \begin{aligned} V_{\mathcal{R}} &= \mathrm{tr}\left[(\mathbf{I} - \tilde{\mathcal{R}})\mathbf{UU}'\right] = \mathrm{tr}\left[(1 - \cos(\varphi))(\mathbf{I} - \boldsymbol{\phi}\boldsymbol{\phi}')\mathbf{UU}'\right] \\ &= (1 - \cos(\varphi))\boldsymbol{\phi}'(\mathrm{tr}(\mathbf{UU}')\mathbf{I} - \mathbf{UU}')\boldsymbol{\phi}. \end{aligned} $$

Using $\|\mathbf{I} - \tilde{\mathcal{R}}\|^2 = 4(1 - \cos(\varphi))$ and the definition of $\mathbf{P}$ produces the desired result. The formulation of V_p is obtained directly by noting that ${}^{B}\hat{\mathbf{u}}_n - {}^{B}\mathbf{u}_n = \hat{\mathbf{p}} - \mathbf{p} = \tilde{\mathbf{p}}$. □

The time derivative of the Lyapunov function along the system trajectories is presented in the following statement.

Lemma 2. *The time derivatives of proposed Lyapunov functions* (12a) *and* (12b) *are respectively given by*

$$ \dot{V}_{\mathcal{R}} = \left(\mathbf{UU}'\tilde{\mathcal{R}} - \tilde{\mathcal{R}}'\mathbf{UU}'\right)'_{\otimes} \mathcal{R}(\hat{\omega} - \omega) \tag{14a} $$

$$ \dot{V}_p = \tilde{\mathbf{p}}'\left((\hat{\mathbf{p}})_\times(\hat{\omega} - \omega) + (\hat{\mathbf{v}} - \mathbf{v})\right). \tag{14b} $$

Proof. Differentiating the Lyapunov function (12b) with respect to time and using (7b) and $\tilde{\mathbf{p}}'(\omega)_\times\tilde{\mathbf{p}} = 0$ yields (14b) directly. Differentiating the Lyapunov function (12a) with respect to time and using (7a) yields

$$\dot{V}_{\mathcal{R}} = -\operatorname{tr}(\dot{\tilde{\mathcal{R}}}\mathbf{U}\mathbf{U}') = -\operatorname{tr}((\mathcal{R}(\hat{\omega} - \omega))_\times \mathbf{U}\mathbf{U}'\tilde{\mathcal{R}}).$$

Using the properties of the trace, presented in Appendix A, produces

$$\begin{aligned}\dot{V}_{\mathcal{R}} &= -\frac{1}{2}\operatorname{tr}((\mathcal{R}(\hat{\omega} - \omega))_\times (\mathbf{U}\mathbf{U}'\tilde{\mathcal{R}} - \tilde{\mathcal{R}}'\mathbf{U}\mathbf{U}')) \\ &= \left(\mathbf{U}\mathbf{U}'\tilde{\mathcal{R}} - \tilde{\mathcal{R}}'\mathbf{U}\mathbf{U}'\right)_\otimes' \mathcal{R}(\hat{\omega} - \omega).\end{aligned}$$

□

The decoupling property of the Lyapunov function allows for the attitude and position estimation problems to be addressed separately. The feedback law for the attitude kinematics (7a) is derived using the Lyapunov function $V_{\mathcal{R}}$, while the feedback law for the position kinematics (7b) relies on V_p.

3.2. Attitude feedback law

The attitude feedback law, derived in this section, exploits angular velocity sensors and landmark measurements. While velocity sensors allow for the propagation of attitude in time, attitude with respect to a reference frame is observed only by means of the landmark measurements. The geometric placement of the landmarks is required to satisfy the following assumption.

Assumption 1 (Landmark Configuration)**.** *The landmarks are not all collinear, that is,* $\operatorname{rank}(\mathbf{X}) \geq 2$.

Assumption 1 specifies the necessary and sufficient landmark configuration under which zero observation error is equivalent to correct attitude estimation, i.e. $\forall_{i=1..n-1} \|{}^B\hat{\mathbf{u}}_i - {}^B\mathbf{u}_i\| = 0 \Leftrightarrow \tilde{\mathcal{R}} = \mathbf{I}$. This is shown in the following proposition, using the fact that the Lyapunov function $V_{\mathcal{R}}$ expresses the estimation error of the transformed landmarks.

Lemma 3. *The Lyapunov function $V_{\mathcal{R}}$, expressed in* (12a)*, has a unique global minimum (at $\tilde{\mathcal{R}} = \mathbf{I}$) if and only if Assumption 1 is verified.*

Proof. From (13), it is straightforward that $V_{\mathcal{R}} \geq 0$. From (12a), the zeros of $V_{\mathcal{R}}$ are $\varphi = 0$ or $\boldsymbol{\phi} \in \mathcal{N}(\mathbf{P})$. To show that $\mathbf{P} > 0$ if and only if $\operatorname{rank}(\mathbf{X}) \geq 2$, denote the singular value decomposition of $\mathbf{U}$ as $\mathbf{U} = \mathbf{U}_U\mathbf{S}_U\mathbf{V}_U'$, where $\mathbf{U}_U \in O(3)$, $\mathbf{V}_U \in O(n)$, the off-diagonal elements of $\mathbf{S}_U \in M(3,n)$ are zero ($\forall_{i\neq j} s_{ij} = 0$) and the diagonal elements are the singular values of $\mathbf{U}$, i.e. $s_{ii} = \sigma_i(\mathbf{U})$, $i \in \{1,2,3\}$. Then $\mathbf{P} = \operatorname{tr}(\mathbf{U}_U\mathbf{U}_U')\mathbf{I} - \mathbf{U}_U\mathbf{U}_U' = \operatorname{tr}(\mathbf{S}_U^2)\mathbf{I} - \mathbf{U}_U\mathbf{S}_U^2\mathbf{U}_U' = \mathbf{U}_U \begin{bmatrix} s_{22}^2+s_{33}^2 & 0 & 0 \\ 0 & s_{11}^2+s_{33}^2 & 0 \\ 0 & 0 & s_{11}^2+s_{22}^2 \end{bmatrix} \mathbf{U}_U'$ and hence $\mathbf{P} > 0$ if and only if $s_{22}, s_{33} \neq 0$, i.e. $\operatorname{rank}(\mathbf{U}) \geq 2$. Given that $\mathbf{A}$ and $\begin{bmatrix}\mathbf{D} & \mathbf{1}_n\end{bmatrix}$ are nonsingular, the equality $\operatorname{rank}(\mathbf{U}) = \operatorname{rank}\left(\begin{bmatrix}\mathbf{U} & \mathbf{0}_3\end{bmatrix}\right) = \operatorname{rank}\left(\mathbf{X}\begin{bmatrix}\mathbf{D} & \mathbf{1}_n\end{bmatrix}\begin{bmatrix}\mathbf{A} & \mathbf{0}_{n-1} \\ \mathbf{0}_{n-1}' & 1\end{bmatrix}\right) = \operatorname{rank}(\mathbf{X})$, completes the proof. □

Remark 1. It is instructive to analyze why a landmark configuration given by $\operatorname{rank}(\mathbf{X}) = 1$ is not sufficient to determine the attitude of the rigid body. If all ${}^L\mathbf{x}_i$ are collinear, then all ${}^L\mathbf{u}_i$ are collinear and any $\tilde{\mathcal{R}} = \operatorname{rot}(\varphi, {}^L\mathbf{u}_i/\|{}^L\mathbf{u}_i\|)$ satisfies ${}^B\hat{\mathbf{u}}_i = {}^B\mathbf{u}_i$, i.e. the estimated and observed landmarks are identical for some $\hat{\mathcal{R}} \neq \mathcal{R}$. This is related to the well known fact that a single vector observation (such as the Earth's magnetic field) yields attitude information except for the rotation about the vector itself [25, 26].

Given the Lyapunov function derivatives along the system trajectories (12a), consider the following feedback law,

$$\hat{\omega} = \omega_r - k_\omega \mathbf{s}_\omega, \tag{15}$$

where the feedback term is given by

$$\mathbf{s}_\omega = \mathcal{R}'\left(\mathbf{U}\mathbf{U}'\tilde{\mathcal{R}} - \tilde{\mathcal{R}}'\mathbf{U}\mathbf{U}'\right)_\otimes, \tag{16}$$

and k_ω is a positive scalar. The attitude feedback yields the autonomous attitude error dynamics

$$\dot{\tilde{\mathcal{R}}} = -k_\omega\tilde{\mathcal{R}}(\mathbf{U}\mathbf{U}'\tilde{\mathcal{R}} - \tilde{\mathcal{R}}'\mathbf{U}\mathbf{U}'), \tag{17}$$

and a negative semidefinite derivative for $V_{\mathcal{R}}$ given by $\dot{V}_{\mathcal{R}} = -k_\omega\mathbf{s}_\omega'\mathbf{s}_\omega \leq 0$. The set where $\dot{V}_{\mathcal{R}} = 0$ is characterized in the following lemma.

Lemma 4. *Under Assumption 1, the set of points where $\dot{V}_{\mathcal{R}} = 0$ is given by*

$$C_{V_{\mathcal{R}}} = \{\tilde{\mathcal{R}} \in \text{SO}(3) : \tilde{\mathcal{R}} = \mathbf{I} \vee \tilde{\mathcal{R}} = \text{rot}(\pi, \boldsymbol{\phi} \in \text{eigvec}(\mathbf{P}))\}.$$

Proof. The points where $\dot{V}_{\mathcal{R}} = 0$ are given by $\mathbf{s}_\omega = 0$. Using Lemma 14 presented in Appendix A, $\mathbf{s}_\omega$ can be rewritten as $\mathbf{s}_\omega = \mathbf{Q}'(\varphi, \boldsymbol{\phi})\mathbf{P}\boldsymbol{\phi}$. The points where $\mathbf{s}_\omega = 0$ satisfy $\mathbf{P}\boldsymbol{\phi} \in \mathcal{N}(\mathbf{Q}'(\varphi, \boldsymbol{\phi}))$, which is equivalent to

$$\sin(\varphi)\mathbf{P}\boldsymbol{\phi} - (1 - \cos(\varphi))\,(\boldsymbol{\phi})_\times\,\mathbf{P}\boldsymbol{\phi} = 0.$$

For any $\mathbf{x} \in \mathbb{R}^3$, $\mathbf{x}$ and $(\boldsymbol{\phi})_\times\,\mathbf{x}$ are noncollinear, and hence $\mathbf{Q}'(\varphi, \boldsymbol{\phi})\mathbf{P}\boldsymbol{\phi} = 0$ if and only if $\varphi = 0$ or $\varphi = \pi$. For the $\varphi = \pi$ case, the equality $(\boldsymbol{\phi})_\times\,\mathbf{P}\boldsymbol{\phi} = 0$ is verified if and only if $\exists_\alpha \mathbf{P}\boldsymbol{\phi} = \alpha\boldsymbol{\phi}$. Consequently, $\dot{V}_{\mathcal{R}} = 0$ if and only if $\varphi = 0 \vee (\varphi = \pi \wedge \boldsymbol{\phi} \in \text{eigvec}(\mathbf{P}))$. □

The open loop dynamics of the Euler angle-axis representation [27] are given by

$$\dot{\varphi} = \boldsymbol{\phi}'\mathcal{R}(\hat{\omega} - \omega),\ \dot{\boldsymbol{\phi}} = \frac{1}{2}\left(\mathbf{I} - \frac{\sin(\varphi)}{1 - \cos(\varphi)}\,(\boldsymbol{\phi})_\times\right)(\boldsymbol{\phi})_\times\,\mathcal{R}(\hat{\omega} - \omega).$$

Using (15), the closed loop dynamics can be written as

$$\dot{\varphi} = -k_\omega \sin(\varphi)\boldsymbol{\phi}'\mathbf{P}\boldsymbol{\phi}, \tag{18a}$$

$$\dot{\boldsymbol{\phi}} = k_\omega\,(\boldsymbol{\phi})_\times\,(\boldsymbol{\phi})_\times\,\mathbf{P}\boldsymbol{\phi}, \tag{18b}$$

where the dynamics of $\boldsymbol{\phi}$ are autonomous.

Evaluating the closed loop dynamics (18) for the points contained in $C_{V_{\mathcal{R}}}$ shows that this set is invariant, which implies that the origin cannot be GAS. The existence of equilibrium points at $\varphi = \pi$ illustrates the topological obstacles to global stabilization when using continuous state feedback for systems defined on manifolds. As discussed in [2, 5, 6], the region of attraction of a stable equilibrium point is homeomorphic to some Euclidean vector space, which precludes global stabilization of the origin on SO(3).

However, the notion of global asymptotic stability can be relaxed by adopting the definition of almost global asymptotic stability (aGAS) [9], in the sense that any trajectory emanating from outside a nowhere dense set of measure zero is attracted to the origin. In the present case, the convergence to the origin $\tilde{\mathcal{R}} = \mathbf{I}$ for all initial conditions outside the set $\varphi = \pi$ can be established. Using the distance on SO(3) inherited by the Euclidean norm, $d(\mathcal{R}_1, \mathcal{R}_2) = ||\mathcal{R}_1 - \mathcal{R}_2||$, the following theorem shows that the origin is aGAS and that the trajectories converge exponentially fast to the desired equilibrium point.

Theorem 5. *The attitude error $\tilde{\mathcal{R}} = \mathbf{I}$ of the closed-loop system* (17) *is aGAS and exponentially stable in any closed ball inside the region of attraction, which is given by*

$$\begin{aligned} R_A &= \{\tilde{\mathcal{R}} \in \text{SO}(3) : ||\mathbf{I} - \tilde{\mathcal{R}}||^2 < 8\} \\ &= \{(\varphi, \boldsymbol{\phi}) \in \text{D}_\phi : \varphi < \pi\}. \end{aligned}$$

where $\text{D}_\phi = [0\ \pi] \times \text{S}(2)$*. For any $\tilde{\mathcal{R}}(t_0) \in R_A$, the solution of the system* (17) *satisfies*

$$||\tilde{\mathcal{R}}(t) - \mathbf{I}|| \leq ||\tilde{\mathcal{R}}(t_0) - \mathbf{I}||e^{-\frac{1}{2}\gamma_{\mathcal{R}}(t - t_0)}, \tag{19}$$

where $\gamma_{\mathcal{R}} = \frac{k_\omega}{4}(8 - ||\tilde{\mathcal{R}}(t_0) - \mathbf{I}||^2)\sigma_3(\mathbf{P}) = k_\omega(1 + \cos(\varphi(t_0)))\sigma_3(\mathbf{P})$.

Proof. Define the Lyapunov function

$$W_{\mathcal{R}} = \frac{||\mathbf{I} - \tilde{\mathcal{R}}||^2}{8} = \frac{1 - \cos(\varphi)}{2}. \tag{20}$$

The time derivative of $W_{\mathcal{R}}$ along the trajectories of the system (17) is described by

$$\dot{W}_{\mathcal{R}} = -k_\omega \frac{||\mathbf{I} - \tilde{\mathcal{R}}||^2}{8}\frac{(8 - ||\mathbf{I} - \tilde{\mathcal{R}}||^2)}{4}\boldsymbol{\phi}'\mathbf{P}\boldsymbol{\phi}.$$

Using $\mathbf{P} > 0$, the set of points where $\dot{W}_{\mathcal{R}} = 0$ is given by $C_W = \{\tilde{\mathcal{R}} \in \text{SO}(3) : \tilde{\mathcal{R}} = \mathbf{I} \vee ||\mathbf{I} - \tilde{\mathcal{R}}||^2 = 8\}$. Since $\dot{W}_{\mathcal{R}} \leq 0$, the set contained in a Lyapunov function surface $\Omega_\rho = \{\tilde{\mathcal{R}} \in \text{SO}(3) : W_{\mathcal{R}} \leq \rho\}$ is positively invariant [28]. Given that $W_{\mathcal{R}} < 1 \Leftrightarrow ||\mathbf{I} - \tilde{\mathcal{R}}||^2 < 8$, the Lyapunov function is strictly decreasing in Ω_ρ for any $\rho < 1$, which implies that

$$||\mathbf{I} - \tilde{\mathcal{R}}(t)||^2 < ||\mathbf{I} - \tilde{\mathcal{R}}(t_0)||^2. \tag{21}$$

for all $t > t_0$ and $||\mathbf{I} - \tilde{\mathcal{R}}(t_0)||^2 < 8$. Rewriting the Lyapunov function time derivative and using (21) yields

$$\dot{W}(\tilde{\mathcal{R}}(t)) = -k_\omega \frac{(8 - ||\mathbf{I} - \tilde{\mathcal{R}}(t)||^2)}{4}\boldsymbol{\phi}'\mathbf{P}\boldsymbol{\phi}W(\tilde{\mathcal{R}}(t)) \Rightarrow$$

$$\dot{W}(\tilde{\mathcal{R}}(t)) \leq -k_\omega \frac{(8 - \|\mathbf{I} - \tilde{\mathcal{R}}(t_0)\|^2)}{4} \sigma_3(\mathbf{P}) W(\tilde{\mathcal{R}}(t)).$$

Applying the comparison lemma [28] and using (20) produces (19), which characterizes the trajectories for the initial conditions $\|\mathbf{I} - \tilde{\mathcal{R}}(t_0)\|^2 < 8$, i.e. $\varphi(t_0) < \pi$.

The closed loop dynamics (18) yield that $\|\mathbf{I} - \tilde{\mathcal{R}}(t_0)\|^2 = 8 \Leftrightarrow \varphi(t_0) = \pi \Rightarrow \dot{\varphi} = 0$ so the set $C_W \setminus \{\mathbf{I}\}$, which is nowhere dense and has measure zero, defines the positively invariant boundary of the region of attraction of $\tilde{\mathcal{R}} = \mathbf{I}$. □

3.3. Position feedback law

The position feedback law is obtained using the methodology adopted for the attitude feedback law derivation. It is immediate that V_p, expressed in (12b), is positive definite, and that $V_p = 0$ if and only if $\tilde{\mathbf{p}} = 0$. Given the time derivative of the Lyapunov function (14b), the position feedback law for the system (7b) is defined as

$$\hat{\mathbf{v}} = \mathbf{v} + ((\omega)_\times - k_v \mathbf{I})\mathbf{s}_v + k_\omega (\hat{\mathbf{p}})_\times \mathbf{s}_\omega, \tag{22}$$

where the feedback term is

$$\mathbf{s}_v = \tilde{\mathbf{p}}, \tag{23}$$

and k_v is a positive scalar. The position feedback law produces a closed loop linear time-invariant system

$$\dot{\tilde{\mathbf{p}}} = -k_v \tilde{\mathbf{p}},$$

whose origin is clearly globally exponentially stable.

For the sake of simplicity, the position was estimated with respect to the landmarks' centroid. Interestingly enough, Appendix B.1 describes how the observer can be formulated to estimate directly the position with respect to the origin of a specific coordinate frame $\{E\}$, translated with respect to $\{L\}$ as illustrated in Fig. 1.

3.4. Sensor based feedback law

In this section, it is shown that the position and attitude feedback laws, (15) and (22) respectively, can be expressed as an explicit function of the velocity measurements (2), landmark readings (3), and observer estimates.

Theorem 6. *The attitude and position feedback laws are explicit functions of the sensor readings and state estimates, and given by*

$$\hat{\omega} = \omega_r - k_\omega \mathbf{s}_\omega, \tag{24a}$$

$$\hat{\mathbf{v}} = \mathbf{v}_r + ((\omega_r)_\times - k_v \mathbf{I})\mathbf{s}_v + k_\omega (\hat{\mathbf{p}})_\times \mathbf{s}_\omega, \tag{24b}$$

$$\mathbf{s}_\omega = \sum_{i=1}^{n} (\hat{\mathcal{R}}'\mathbf{XDAe}_i) \times (\mathbf{QDAe}_i), \tag{24c}$$

$$\mathbf{s}_v = \hat{\mathbf{p}} + \frac{1}{n}\sum_{i=1}^{n} \mathbf{q}_i. \tag{24d}$$

Proof. The formulation of the feedback terms $\hat{\omega}$ and $\hat{\mathbf{v}}$ is obtained directly from (2), (15), and (22). Using the landmark measurement formulation (3) produces

$$\mathbf{s}_v = \hat{\mathbf{p}} + \frac{1}{n}\sum_{i=1}^{n} \mathbf{q}_i = \hat{\mathbf{p}} + \frac{1}{n}\sum_{i=1}^{n} ({}^L\mathbf{x}_i - \mathbf{p}) = \hat{\mathbf{p}} - \mathbf{p} - \frac{1}{n}\sum_{i=1}^{n} {}^L\mathbf{x}_i.$$

Applying the property (4) bears $\mathbf{s}_v = \tilde{\mathbf{p}}$, as desired.

Using the properties of the skew and unskew operators, presented in Appendix A, yields

$$\begin{aligned}\mathbf{s}_\omega &= \mathcal{R}'\left(\mathbf{UU}'\tilde{\mathcal{R}} - \tilde{\mathcal{R}}'\mathbf{UU}'\right)_\otimes = \left(\mathcal{R}'\mathbf{UU}'\hat{\mathcal{R}} - \hat{\mathcal{R}}'\mathbf{UU}'\mathcal{R}\right)_\otimes \\ &= \left({}^B\mathbf{U}{}^B\hat{\mathbf{U}}' - {}^B\hat{\mathbf{U}}{}^B\mathbf{U}'\right)_\otimes = \left(\sum_{i=1}^{n}\left({}^B\mathbf{u}_i{}^B\hat{\mathbf{u}}_i' - {}^B\hat{\mathbf{u}}_i{}^B\mathbf{u}_i'\right)\right)_\otimes \\ &= \left(\sum_{i=1}^{n}\left({}^B\hat{\mathbf{u}}_i \times {}^B\mathbf{u}_i\right)_\times\right)_\otimes = \sum_{i=1}^{n}\left({}^B\hat{\mathbf{u}}_i \times {}^B\mathbf{u}_i\right).\end{aligned}$$

Expanding ${}^B\mathbf{u}_i$ and ${}^B\hat{\mathbf{u}}_i$ using (9) and (10), respectively, produces ${}^B\mathbf{u}_i = {}^B\mathbf{U}\mathbf{e}_i = \mathbf{QDAe}_i$, and ${}^B\hat{\mathbf{u}}_i = {}^B\hat{\mathbf{U}}\mathbf{e}_i = \hat{\mathcal{R}}'\mathbf{XDAe}_i$, which concludes the proof. □

3.5. Directionality of the observer estimates

This section shows that the solutions of the system (17) are influenced by the adopted landmark transformation and the associated matrix $\mathbf{P}$. The trajectories of the observer estimates are characterized using the Euler angle-axis parametrization.

Theorem 7. *Consider the system* (18)*, and let the singular values of* $\mathbf{P}$ *satisfy* $\sigma_1(\mathbf{P}) > \sigma_2(\mathbf{P}) > \sigma_3(\mathbf{P})$*. The attitude error angle* φ *decreases monotonically in* R_A*, and the asymptotic convergence of the Euler axis is described by*

$$\begin{cases} \lim_{t\to\infty} \boldsymbol{\phi}(t) = \operatorname{sign}(\mathbf{n}_3'\boldsymbol{\phi}(t_0))\mathbf{n}_3, & \textit{if } \mathbf{n}_3'\boldsymbol{\phi}(t_0) \neq 0 \\ \lim_{t\to\infty} \boldsymbol{\phi}(t) \in \{\mathbf{n}_1, \mathbf{n}_2\}, & \textit{if } \mathbf{n}_3'\boldsymbol{\phi}(t_0) = 0 \end{cases},$$

where $\mathbf{n}_i$ is the unit eigenvector of $\mathbf{P}$ *associated with* $\sigma_i(\mathbf{P})$.

Proof. The Lyapunov function (20) is strictly decreasing, which implies that the attitude error is monotonically decreasing in the region of attraction R_A. To analyze the convergence of the rotation vector dynamics (18b), define the Lyapunov functions

$$V_s = 1 + s\mathbf{n}_3'\boldsymbol{\phi}, \qquad \dot{V}_s = s\mathbf{n}_3'\boldsymbol{\phi}\left(\boldsymbol{\phi}'\mathbf{P}\boldsymbol{\phi} - \sigma_3(\mathbf{P})\right), \quad (25)$$

in the domain S(2), where $s \in \{-1, 1\}$. From the Schwartz inequality, the Lyapunov function is positive definite and $V_s = 0 \Leftrightarrow \boldsymbol{\phi} = -s\mathbf{n}_3$. Assuming that the eigenvalue has multiplicity 1, the set of points where $\dot{V}_s = 0$ is given by $C_{V_s} = \{\boldsymbol{\phi} \in S(2) : \boldsymbol{\phi} = \pm\mathbf{n}_3 \vee \mathbf{n}_3'\boldsymbol{\phi} = 0\}$. The Lyapunov time derivatives $\dot{V}_{s=-1}$ and $\dot{V}_{s=1}$ are indefinite in the domain S(2). For each initial condition $\boldsymbol{\phi}(t_0)$ choose s and $0 < \beta < 1$ such that $s\mathbf{n}_3'\boldsymbol{\phi}(t_0) \leq \beta - 1 < 0$, i.e. $V_s(\boldsymbol{\phi}(t_0)) \leq \beta$. The level sets $\Omega_\beta^s = \{\boldsymbol{\phi} \in S(2) : V_s(\boldsymbol{\phi}) \leq \beta\}$, are positively invariant. The unique points where $\dot{V}_s = 0$ in Ω_β^s, given by $\boldsymbol{\phi} = -s\mathbf{n}_3$, are asymptotically stable.

To analyze the case $\mathbf{n}_3'\boldsymbol{\phi}(t_0) = 0$, the property $\mathbf{n}_3'\boldsymbol{\phi} = 0 \Rightarrow \mathbf{n}_3'\dot{\boldsymbol{\phi}} = 0$ shows that the set defined by $\mathbf{n}_3'\boldsymbol{\phi} = 0$ is positively invariant, and hence $\boldsymbol{\phi}(t) \in \text{span}(\mathbf{n}_1, \mathbf{n}_2)$ for all t. Using Lemma 4 implies that $\boldsymbol{\phi}(t) \to \{\mathbf{n}_1, \mathbf{n}_2\}$ as $t \to \infty$. □

The asymptotic convergence for the specific case $\exists_{i\neq j}\sigma_i(\mathbf{P}) = \sigma_j(\mathbf{P})$ can be obtained by following the same steps of the proof of Theorem 7. In particular, if $\exists_\sigma \mathbf{P} = \sigma\mathbf{I}$, then every point $\boldsymbol{\phi}(t) \in S(2)$ is stable.

Proposition 8. *Let* $\exists_{i\neq j}\sigma_i(\mathbf{P}) = \sigma_j(\mathbf{P})$. *The asymptotic convergence of the attitude error dynamics is characterized as follows.*

- *If* $\sigma_1(\mathbf{P}) = \sigma_2(\mathbf{P}) > \sigma_3(\mathbf{P})$, *then*

$$\begin{cases} \lim_{t\to\infty} \boldsymbol{\phi}(t) = \text{sign}(\mathbf{n}_3'\boldsymbol{\phi}(t_0))\mathbf{n}_3, & \text{if } \mathbf{n}_3'\boldsymbol{\phi}(t_0) \neq 0 \\ \boldsymbol{\phi}(t) = \boldsymbol{\phi}(t_0) & \text{if } \mathbf{n}_3'\boldsymbol{\phi}(t_0) = 0 \end{cases}, \quad (26)$$

- *If* $\sigma_1(\mathbf{P}) > \sigma_2(\mathbf{P}) = \sigma_3(\mathbf{P})$, *then*

$$\begin{cases} \lim_{t\to\infty} \boldsymbol{\phi}(t) = \text{span}(\mathbf{n}_2, \mathbf{n}_3), & \text{if } \boldsymbol{\phi}(t_0) \neq \pm\mathbf{n}_1 \\ \boldsymbol{\phi}(t) = \boldsymbol{\phi}(t_0), & \text{if } \boldsymbol{\phi}(t_0) = \pm\mathbf{n}_1 \end{cases} \quad (27)$$

- *If* $\sigma_1(\mathbf{P}) = \sigma_2(\mathbf{P}) = \sigma_3(\mathbf{P})$, *then* $\boldsymbol{\phi}(t) = \boldsymbol{\phi}(t_0)$.

Proof. The asymptotic convergence (26) results directly from Theorem 7 and from $\mathbf{n}_3'\boldsymbol{\phi}(t) = 0 \Rightarrow \dot{\boldsymbol{\phi}}(t) = 0$. The asymptotic convergence (27) is obtained using the Lyapunov functions (25). The set of points where $\dot{V}_s = 0$ is given by $C_{V_s} = \{\boldsymbol{\phi} \in S(2) : \boldsymbol{\phi} \in \text{span}(\mathbf{n}_2, \mathbf{n}_3) \vee \mathbf{n}_3'\boldsymbol{\phi} = 0\}$, and hence, by LaSalle's principle, $\boldsymbol{\phi} \to \text{span}(\mathbf{n}_2, \mathbf{n}_3)$ as $t \to \infty$ if $s\mathbf{n}_3'\boldsymbol{\phi}(t_0) < 0$, i.e. $\mathbf{n}_3'\boldsymbol{\phi}(t_0) \neq 0$. Using the Lyapunov functions $V_s(\boldsymbol{\phi}) = 1 + s\mathbf{n}_2'\boldsymbol{\phi}$, and LaSalle's principle, bears $\boldsymbol{\phi} \to \text{span}(\mathbf{n}_2, \mathbf{n}_3)$ as $t \to \infty$ if $\mathbf{n}_2'\boldsymbol{\phi}(t_0) \neq 0$. Using the kinematics (18b), it is immediate that $\text{span}(\mathbf{n}_2, \mathbf{n}_3)$ and $\{-\mathbf{n}_1, \mathbf{n}_1\}$ are positively invariant sets. □

The results of Theorem 7 and Proposition 8 show that, for almost all initial conditions, $\boldsymbol{\phi}$ converges to the direction of the smallest singular value of $\mathbf{P}$. This characterization of the attitude error is of interest in navigation system design, allowing the system designer to shape the fastest and slowest directions of estimation using the landmark coordinate transformation (8).

4. Observer synthesis with biased velocity readings

In this section, the observer architecture is extended to compensate for the bias in the linear and angular velocity readings (2), where $\mathbf{b}_\omega$ and $\mathbf{b}_v$ are unknown. The derived attitude and position feedback laws bear coupled, non-autonomous position and attitude error kinematics, and hence the stability of the resulting observer is analyzed using a single Lyapunov function.

The proposed Lyapunov function (11) is augmented to account for the effect of the angular and linear velocity bias

$$\begin{aligned} V_b &= \frac{1}{\gamma_\varphi}V_{\mathcal{R}} + \frac{1}{\gamma_p}V_p + \frac{\gamma_{b_\omega}}{2}\|\tilde{\mathbf{b}}_\omega\|^2 + \frac{\gamma_{b_v}}{2}\|\tilde{\mathbf{b}}_v\|^2 \\ &= \frac{\gamma_\varphi}{4}\|\mathbf{I} - \tilde{\mathcal{R}}\|^2\boldsymbol{\phi}'\mathbf{P}\boldsymbol{\phi} + \frac{\gamma_p}{2}\|\tilde{\mathbf{p}}\|^2 + \frac{\gamma_{b_\omega}}{2}\|\tilde{\mathbf{b}}_\omega\|^2 + \frac{\gamma_{b_v}}{2}\|\tilde{\mathbf{b}}_v\|^2 \end{aligned} \quad (28)$$

where $\tilde{\mathbf{b}}_\omega = \hat{\mathbf{b}}_\omega - \mathbf{b}_\omega$, $\tilde{\mathbf{b}}_v = \hat{\mathbf{b}}_v - \mathbf{b}_v$ are the bias compensation errors, $\hat{\mathbf{b}}_\omega$, $\hat{\mathbf{b}}_v$ are the estimated biases, and γ_φ, γ_p, γ_{b_ω} and γ_{b_v} are positive scalars.

Under Assumption 1 and given the result of Lemma 3, the Lyapunov function V_b has an unique global minimum at

$(\tilde{\mathbf{p}}, \tilde{\mathcal{R}}, \tilde{\mathbf{b}}_\omega, \tilde{\mathbf{b}}_v) = (\mathbf{0}, \mathbf{I}, \mathbf{0}, \mathbf{0})$. The feedback law for biased velocity readings is designed by shaping $\mathbf{P}$ with uniform directionality, using the transformation $\mathbf{A}$.

Proposition 9. *Let* $\mathbf{H} := \mathbf{XD}$ *be full rank, there is a nonsingular* $\mathbf{A} \in \mathrm{M}(n)$ *such that* $\mathbf{UU}' = \mathbf{I}$.

Proof. Take the singular value decomposition of $\mathbf{H} = \mathbf{U}_H\mathbf{S}_H\mathbf{V}_H'$ where $\mathbf{U}_H \in \mathrm{O}(3)$, $\mathbf{V}_H \in \mathrm{O}(n)$, $\mathbf{S}_H = \begin{bmatrix}\mathrm{diag}(s_1, s_2, s_3) & \mathbf{0}_{3\times(n-3)}\end{bmatrix} \in \mathrm{M}(3, n)$, and $s_1 \geq s_2 \geq s_3 > 0$ are the singular values of $\mathbf{H}$. Any $\mathbf{A}$ given by $\mathbf{A} = \mathbf{V}_H \begin{bmatrix}\mathrm{diag}(s_1^{-1}, s_2^{-1}, s_3^{-1}) & \mathbf{0}_{3\times(n-3)} \\ \mathbf{0}_{(n-3)\times 3} & \mathbf{B}\end{bmatrix} \mathbf{V}_A'$, where $\mathbf{B} \in \mathrm{M}(n-3)$ is nonsingular and $\mathbf{V}_A \in \mathrm{O}(n)$, produces $\mathbf{UU}' = \mathbf{HAA}'\mathbf{H} = \mathbf{U}_H\mathbf{V}_A'\mathbf{V}_A\mathbf{U}_H' = \mathbf{I}$. □

Remark 2. Given that $\mathrm{rank}(\mathbf{H}) = \mathrm{rank}(\mathbf{X})$, the condition $\mathrm{rank}(\mathbf{X}) = 2$ of Assumption 1 does not satisfy directly the conditions of Proposition 9. In that case, the observer equations can be rewritten, by taking two linearly independent columns of $\mathbf{H}$, ${}^L\mathbf{h}_i$ and ${}^L\mathbf{h}_j$, an constructing a full rank matrix, $\mathbf{H}_a = \begin{bmatrix}\mathbf{H} & {}^L\mathbf{h}_i \times {}^L\mathbf{h}_j\end{bmatrix}$. This procedure is discussed in detail in Appendix B.2.

Using the transformation $\mathbf{A}$ defined in Proposition 9, the Lyapunov function expressed in (28) is given by

$$V_b = \frac{\gamma_\varphi}{2}\|\mathbf{I} - \tilde{\mathcal{R}}\|^2 + \frac{\gamma_p}{2}\|\tilde{\mathbf{p}}\|^2 + \frac{\gamma_{b_\omega}}{2}\|\tilde{\mathbf{b}}_\omega\|^2 + \frac{\gamma_{b_v}}{2}\|\tilde{\mathbf{b}}_v\|^2. \quad (29)$$

The time derivative of the Lyapunov function results in

$$\begin{aligned}\dot{V}_b = {} & \gamma_p\mathbf{s}_v' \left((\hat{\mathbf{p}})_\times (\hat{\omega} - \omega) + (\hat{\mathbf{v}} - \mathbf{v}) - (\omega)_\times \tilde{\mathbf{p}}\right) \\ & + \gamma_\varphi \mathbf{s}_\omega'(\hat{\omega} - \omega) + \gamma_{b_\omega}\tilde{\mathbf{b}}_\omega'\dot{\tilde{\mathbf{b}}}_\omega + \gamma_{b_v}\tilde{\mathbf{b}}_v'\dot{\tilde{\mathbf{b}}}_v, \quad (30)\end{aligned}$$

where $\mathbf{s}_\omega$ and $\mathbf{s}_v$ are given by (16) and (23), and by considering the transformation $\mathbf{A}$ formulated in Proposition 9, that is

$$\mathbf{s}_\omega = \mathcal{R}'\left(\tilde{\mathcal{R}} - \tilde{\mathcal{R}}'\right)_\otimes, \quad \mathbf{s}_v = \tilde{\mathbf{p}}. \quad (31)$$

The feedback laws for the angular and linear velocities are obtained by rewriting (24a) and (24b) respectively, with compensation of the velocity sensors bias, producing

$$\hat{\omega} = (\omega_r - \hat{\mathbf{b}}_\omega) - k_\omega\mathbf{s}_\omega = (\omega - \tilde{\mathbf{b}}_\omega) - k_\omega\mathbf{s}_\omega, \quad (32a)$$

$$\begin{aligned}\hat{\mathbf{v}} &= \mathbf{v}_r - \hat{\mathbf{b}}_v + \left(\left(\omega_r - \hat{\mathbf{b}}_\omega\right)_\times - k_v\mathbf{I}\right)\mathbf{s}_v - (\hat{\mathbf{p}})_\times (\hat{\omega} - (\omega_r - \hat{\mathbf{b}}_\omega)) \\ &= \mathbf{v} - \tilde{\mathbf{b}}_v + \left(\left(\omega - \tilde{\mathbf{b}}_\omega\right)_\times - k_v\mathbf{I}\right)\mathbf{s}_v + k_\omega (\hat{\mathbf{p}})_\times \mathbf{s}_\omega. \quad (32b)\end{aligned}$$

Using the feedback terms $\hat{\omega}$ and $\hat{\mathbf{v}}$ in (30) yields

$$\begin{aligned}\dot{V}_b = {} & -\gamma_p k_v\|\mathbf{s}_v\|^2 - \gamma_\varphi k_\omega\|\mathbf{s}_\omega\|^2 \\ & + (\gamma_p (\hat{\mathbf{p}})_\times \tilde{\mathbf{p}} - \gamma_\varphi\mathbf{s}_\omega + \gamma_{b_\omega}\dot{\tilde{\mathbf{b}}}_\omega)'\tilde{\mathbf{b}}_\omega + (\gamma_{b_v}\dot{\tilde{\mathbf{b}}}_v - \gamma_p\mathbf{s}_v)'\tilde{\mathbf{b}}_v.\end{aligned}$$

The bias estimates satisfy $\dot{\hat{\mathbf{b}}}_\omega = \dot{\tilde{\mathbf{b}}}_\omega$, $\dot{\hat{\mathbf{b}}}_v = \dot{\tilde{\mathbf{b}}}_v$, and the bias feedback laws are defined as

$$\dot{\hat{\mathbf{b}}}_\omega = \frac{1}{\gamma_{b_\omega}}\left(\gamma_\varphi\mathbf{s}_\omega - \gamma_p (\hat{\mathbf{p}})_\times \mathbf{s}_v\right), \quad \dot{\hat{\mathbf{b}}}_v = \frac{\gamma_p}{\gamma_{b_v}}\mathbf{s}_v,$$

producing the Lyapunov function time derivative $\dot{V}_b = -\gamma_p k_v\mathbf{s}_v'\mathbf{s}_v - \gamma_\varphi k_\omega\mathbf{s}_\omega'\mathbf{s}_\omega$, that is negative semi-definite.

The dynamics of the closed-loop estimation errors are described by

$$\dot{\tilde{\mathbf{p}}} = -(\mathbf{p})_\times \tilde{\mathbf{b}}_\omega - k_v\tilde{\mathbf{p}} - \tilde{\mathbf{b}}_v, \; \dot{\tilde{\mathcal{R}}} = -k_\omega\tilde{\mathcal{R}}(\tilde{\mathcal{R}} - \tilde{\mathcal{R}}') - \tilde{\mathcal{R}}\left(\mathcal{R}\tilde{\mathbf{b}}_\omega\right)_\times, \quad (33a)$$

$$\dot{\tilde{\mathbf{b}}}_\omega = \frac{\gamma_\varphi}{\gamma_{b_\omega}}\mathcal{R}\left(\tilde{\mathcal{R}} - \tilde{\mathcal{R}}'\right)_\otimes - \frac{\gamma_p}{\gamma_{b_\omega}}(\mathbf{p})_\times \tilde{\mathbf{p}}, \quad \dot{\tilde{\mathbf{b}}}_v = \frac{\gamma_p}{\gamma_{b_v}}\tilde{\mathbf{p}}. \quad (33b)$$

The system (33) is nonautonomous, and the compensation of rate gyro bias couples the attitude and position dynamics.

To analyze the stability of (33), define the state $\mathbf{x}_b = (\tilde{\mathbf{p}}, \tilde{\mathcal{R}}, \tilde{\mathbf{b}}_\omega, \tilde{\mathbf{b}}_p)$ and the domain $D_b = \mathbb{R}^3 \times \mathrm{SO}(3) \times \mathbb{R}^3 \times \mathbb{R}^3$, the set of points where $\dot{V}_b = 0$ is given by

$$\begin{aligned}C_{V_b} &= \{\mathbf{x}_b \in D_b : (\tilde{\mathbf{p}}, \tilde{\mathbf{b}}_\omega, \tilde{\mathbf{b}}_p) = (\mathbf{0}, \mathbf{0}, \mathbf{0}), \tilde{\mathcal{R}} \in C_{\mathcal{R}}\}, \\ C_{\mathcal{R}} &= \{\tilde{\mathcal{R}} \in \mathrm{SO}(3) : \tilde{\mathcal{R}} = \mathbf{I} \vee \tilde{\mathcal{R}} = \mathrm{rot}(\pi, \boldsymbol{\phi} \in \mathrm{S}(2))\}.\end{aligned}$$

In the next proposition, the boundedness of the estimation errors is shown and used to provide sufficient conditions for excluding convergence to the equilibrium points $\tilde{\mathcal{R}} = \mathrm{rot}(\pi, \boldsymbol{\phi})$.

Lemma 10. *The estimation errors* $(\tilde{\mathbf{p}}, \tilde{\mathcal{R}}, \tilde{\mathbf{b}}_\omega, \tilde{\mathbf{b}}_p)$ *are bounded. For any initial condition such that*

$$\frac{\gamma_{b_v}\|\tilde{\mathbf{b}}_v(t_0)\|^2 + \gamma_p\|\tilde{\mathbf{p}}(t_0)\|^2 + \gamma_{b_\omega}\|\tilde{\mathbf{b}}_\omega(t_0)\|^2}{\gamma_\varphi(8 - \|\mathbf{I} - \tilde{\mathcal{R}}(t_0)\|^2)} < 1, \quad (34)$$

the attitude error is bounded by $\|\mathbf{I} - \tilde{\mathcal{R}}(t)\|^2 \leq c_{\max} < 8$ *for all* $t \geq t_0$.

Proof. Define the set $\Omega_\rho = \{\mathbf{x}_b \in \mathrm{D}_b : V_b \leq \rho\}$. The Lyapunov function (29) is the weighted distance of the state to the origin, so $\exists_\alpha \|\mathbf{x}_b\|^2 \leq \alpha V_b$ and the set Ω_ρ is compact. The Lyapunov function decreases along the system trajectories, $\dot{V}_b \leq 0$, so any trajectory starting in Ω_ρ will remain in Ω_ρ and satisfy $V_b(\mathbf{x}_b(t)) \leq V_b(\mathbf{x}_b(t_0))$. Consequently, $\forall_{t\geq t_0} \|\mathbf{x}_b(t)\|^2 \leq \alpha V_b(\mathbf{x}(t_0))$ and the state is bounded.

The gain condition (34) is equivalent to $V_b(\mathbf{x}_b(t_0)) \leq \gamma_\varphi(4 - \varepsilon)$ for some ε sufficiently small. Using $V_b(\mathbf{x}_b(t)) \leq V_b(\mathbf{x}_b(t_0))$ implies that $\gamma_\varphi \|\mathbf{I} - \tilde{\mathcal{R}}(t)\|^2 \leq 2V_b(\mathbf{x}_b(t_0))$ for all $t \geq t_0$, hence choosing $c_{\max} = 8 - 2\varepsilon$ concludes the proof. □

Remark 3. The formulation of Lemma 10 can be expressed as a function of the rotation error φ, which is a scalar quantity and hence provides for a more intuitive representation of the bounds. The inequality (34) can be rewritten as

$$\frac{\gamma_{b_v}\|\tilde{\mathbf{b}}_v(t_0)\|^2 + \gamma_p\|\tilde{\mathbf{p}}(t_0)\|^2 + \gamma_{b_\omega}\|\tilde{\mathbf{b}}_\omega(t_0)\|^2}{4\gamma_\varphi(1+\cos(\varphi(t_0)))} < 1,$$

and the bound $\|\mathbf{I} - \tilde{\mathcal{R}}(t)\|^2 \leq c_{\max} < 8$ is equivalent to $\varphi(t) \leq \varphi_{\max} < \pi$, where $\varphi_{\max} = \arccos(1 - \frac{c_{\max}}{4})$.

Adopting the analysis tools for parameterized LTV systems [22], the system (33), in the form $\dot{\mathbf{x}}_b = f(t, \mathbf{x}_b)\mathbf{x}_b$, is rewritten as $\dot{\mathbf{x}}_\star = \mathbf{A}(\lambda, t)\mathbf{x}_\star$. In this formulation, the parameter $\lambda \in D_b \times \mathbb{R}$ is associated with the initial conditions of the nonlinear system and the solutions of both systems are identical whenever the initial conditions of both systems coincide, $\mathbf{x}_\star(t_0) = \mathbf{x}(t_0)$, and the parameter satisfies $\lambda = (t_0, \mathbf{x}(t_0))$.

The results derived in [22] establish sufficient conditions for exponential stability of the parameterized LTV system, uniformly in the parameter λ (λ-UGES). As discussed in [22], λ-UGES of the parameterized LTV system implies that the origin of the associated nonlinear system is exponentially stable, see Appendix C for more details. These results are used to show that the estimation errors in the bounded set (34) converge exponentially fast to the origin in the presence of biased velocity measurements.

Theorem 11. *Let $\gamma_{b_v} = \gamma_{b_\omega}$ and assume that $\mathbf{p}$, $\mathbf{v}$, and ω are bounded. For any initial condition that satisfies* (34)*, the position, attitude and bias estimation errors converge exponentially fast to the stable equilibrium point* $(\tilde{\mathbf{p}}, \tilde{\mathcal{R}}, \tilde{\mathbf{b}}_\omega, \tilde{\mathbf{b}}_v) = (0, \mathbf{I}, 0, 0)$.

Proof. The stability of (33) is obtained by a change of coordinates to the quaternion form. Let the attitude error vector be given by $\tilde{\mathbf{q}}_q = \frac{(\tilde{\mathcal{R}}-\tilde{\mathcal{R}}')_\otimes}{\|(\tilde{\mathcal{R}}-\tilde{\mathcal{R}}')_\otimes\|}\frac{\|\mathbf{I}-\tilde{\mathcal{R}}\|}{2\sqrt{2}}$, the closed loop kinematics are described by

$$\dot{\tilde{\mathbf{p}}} = -(\mathbf{p})_\times\tilde{\mathbf{b}}_\omega - k_v\tilde{\mathbf{p}} - \tilde{\mathbf{b}}_v,\ \dot{\tilde{\mathbf{q}}}_q = \frac{1}{2}\mathbf{Q}(\tilde{\mathbf{q}})(-\mathcal{R}\tilde{\mathbf{b}}_\omega - 4k_\omega\tilde{\mathbf{q}}_q\tilde{q}_s), \tag{35a}$$

$$\dot{\tilde{\mathbf{b}}}_\omega = 4\frac{\gamma_\varphi}{\gamma_{b_\omega}}\mathcal{R}'\mathbf{Q}'(\tilde{\mathbf{q}})\tilde{\mathbf{q}}_q - \frac{\gamma_p}{\gamma_{b_\omega}}(\mathbf{p})_\times\tilde{\mathbf{p}},\ \dot{\tilde{\mathbf{b}}}_v = \frac{\gamma_p}{\gamma_{b_v}}\tilde{\mathbf{p}}, \tag{35b}$$

where $\mathbf{Q}(\tilde{\mathbf{q}}) := \tilde{q}_s\mathbf{I} + \left(\tilde{\mathbf{q}}_q\right)_\times$, $\tilde{\mathbf{q}} = \begin{bmatrix}\tilde{\mathbf{q}}_q' & \tilde{q}_s\end{bmatrix}'$, $\tilde{q}_s = \frac{1}{2}\sqrt{1+\mathrm{tr}(\tilde{\mathcal{R}})}$ and $\dot{\tilde{q}}_s = 2k_\omega\tilde{\mathbf{q}}_q'\tilde{\mathbf{q}}_q\tilde{q}_s - \frac{1}{2}\mathbf{q}_q'\tilde{\mathbf{b}}_\omega$. The vector $\tilde{\mathbf{q}}$ is the well known Euler quaternion representation [24]. Using $\|\tilde{\mathbf{q}}_q\|^2 = \frac{1}{8}\|\tilde{\mathcal{R}} - \mathbf{I}\|^2$, the Lyapunov function in quaternion coordinates is described by $V_b = 4\gamma_\varphi\|\tilde{\mathbf{q}}_q\|^2 + \frac{\gamma_p}{2}\|\tilde{\mathbf{p}}\|^2 + \frac{\gamma_{b_\omega}}{2}\|\tilde{\mathbf{b}}_\omega\|^2 + \frac{\gamma_{b_v}}{2}\|\tilde{\mathbf{b}}_v\|^2$.

Let $\mathbf{x}_q := (\tilde{\mathbf{p}}, \tilde{\mathbf{q}}_q, \tilde{\mathbf{b}}_\omega, \tilde{\mathbf{b}}_v)$, $\mathbf{x}_q \in D_q$, and $D_q := \mathbb{R}^3\times\mathrm{B}(3)\times\mathbb{R}^3\times\mathbb{R}^3$, define the system (35) in the domain $\mathcal{D}_q = \{\mathbf{x} \in D_q : V_b \leq \gamma_\varphi(4-\varepsilon_q)\}$, $0 < \varepsilon_q < 4$. The set $\mathcal{D}_q$ corresponds to the interior of the Lyapunov surface, so it is positively invariant and well defined. The condition (34) implies that the initial condition is contained in the set $\mathcal{D}_q$ for ε_q small enough and, by Lemma 10, the components of the attitude error quaternion are bounded by $\|\tilde{\mathbf{q}}_q\|^2 \leq \frac{c_{\max}}{8}$ and $\|\tilde{q}_s\|^2 \geq 1 - \frac{c_{\max}}{8}$, with $c_{\max} = 8 - 2\varepsilon_q$.

Let $\mathbf{x}_\star := (\tilde{\mathbf{p}}_\star, \tilde{\mathbf{q}}_{q\star}, \tilde{\mathbf{b}}_{\omega\star}, \tilde{\mathbf{b}}_{v\star})$, $D_q := \mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3\times\mathbb{R}^3$, $\gamma_b := \gamma_{b_\omega} = \gamma_{b_v}$, and define the parameterized LTV system

$$\dot{\mathbf{x}}_\star = \begin{bmatrix}\mathcal{A}(t,\lambda) & \mathcal{B}'(t,\lambda)\\ -C(t,\lambda) & \mathbf{0}_{3\times3}\end{bmatrix}\mathbf{x}_\star, \tag{36}$$

where $\lambda \in \mathbb{R}_{\geq 0}\times\mathcal{D}_q$, the submatrices are described by

$$\mathcal{A}(t,\lambda) = \begin{bmatrix}-k_v\mathbf{I} & \mathbf{0}_{3\times3}\\ \mathbf{0}_{3\times3} & -2k_\omega\tilde{q}_s(t,\lambda)\mathbf{Q}(\tilde{\mathbf{q}}(t,\lambda))\end{bmatrix},$$

$$\mathcal{B}(t,\lambda) = \begin{bmatrix}(\mathbf{p})_\times & -\frac{\mathcal{R}'\mathbf{Q}'(\tilde{\mathbf{q}}(t,\lambda))}{2}\\ -\mathbf{I} & \mathbf{0}_{3\times3}\end{bmatrix},\ C(t,\lambda) = \frac{\mathcal{B}(t,\lambda)}{\gamma_b}\begin{bmatrix}\gamma_p\mathbf{I} & 0\\ 0 & 8\gamma_\varphi\mathbf{I}\end{bmatrix},$$

and the quaternion $\tilde{\mathbf{q}}(t,\lambda)$ represents the solution of (35) with initial condition $\lambda = (t_0, \tilde{\mathbf{p}}(t_0), \tilde{\mathbf{q}}_q(t_0), \tilde{\mathbf{b}}_\omega(t_0), \tilde{\mathbf{b}}_v(t_0))$. By the boundedness of $\mathbf{p}$, the matrices $\mathcal{A}(t,\lambda)$, $\mathcal{B}(t,\lambda)$ and $C(t,\lambda)$ are bounded, and the system is well defined [28]. If the parameterized LTV (36) is λ-UGES, then the nonlinear system (35) is uniformly exponentially stable in the domain $\mathcal{D}_q$, see Appendix C for details. The parameterized LTV system verifies the assumptions of [22, Theorem 1]:

1) Given the boundedness of $\mathbf{p}$, $\mathbf{v}$ and ω, $\dot{\mathbf{p}}$ is bounded, and the elements of $\mathcal{B}(t,\lambda)$ and $\frac{\partial \mathcal{B}(t,\lambda)}{\partial t} = \begin{bmatrix} ({}^B\dot{\mathbf{p}})_\times & -\frac{1}{2}\dot{\mathcal{R}}'\mathbf{Q}'(\tilde{\mathbf{q}}(t,\lambda))+\mathcal{R}'\mathbf{Q}'(\dot{\tilde{\mathbf{q}}}(t,\lambda)) \\ \mathbf{0}_{3\times3} & \mathbf{0}_{3\times3} \end{bmatrix}$, as well as the corresponding induced Euclidean norm, are bounded for all $\lambda \in \mathbb{R}_{\geq 0}\times\mathcal{D}_q$, $t \geq t_0$.

2) The positive definite matrices $\mathbf{P}(t,\lambda) = \begin{bmatrix} \frac{\gamma_p}{\gamma_b}\mathbf{I} & 0 \\ 0 & 8\frac{\gamma_\varphi}{\gamma_b}\mathbf{I} \end{bmatrix}$ and $\mathbf{Q}(t,\lambda) = \begin{bmatrix} 2k_v\frac{\gamma_p}{\gamma_b}\mathbf{I} & 0 \\ 0 & 32\tilde{q}_s^2(t,\lambda)k_\omega\frac{\gamma_\varphi}{\gamma_b}\mathbf{I} \end{bmatrix}$, satisfy $\mathbf{P}(t,\lambda)\mathcal{B}'(t,\lambda) = C'(t,\lambda)$, $-\mathbf{Q}(t,\lambda) = \mathcal{A}'(t,\lambda)\mathbf{P}(t,\lambda) + \mathbf{P}(t,\lambda)\mathcal{A}(t,\lambda) + \dot{\mathbf{P}}(t,\lambda)$, $\min(C_P)\mathbf{I} \leq \mathbf{P}(t,\lambda) \leq \max(C_P)\mathbf{I}$, $\min(C_Q)\mathbf{I} \leq \mathbf{Q}(t,\lambda) \leq \max(C_Q)\mathbf{I}$, with $C_P = \frac{1}{\gamma_b}\{\gamma_p, 8\gamma_\varphi\}$ and $C_Q = \frac{1}{\gamma_b}\{32k_\omega\gamma_\varphi, 32k_\omega\gamma_\varphi(1-\frac{c_{\max}}{8}), 2k_v\gamma_p\}$.

The system (36) is λ-UGES if and only if $\mathcal{B}(t,\lambda)$ is λ-uniformly persistently exciting [22]. Algebraic manipulation produces $\mathcal{B}(\tau,\lambda)\mathcal{B}'(\tau,\lambda) = \begin{bmatrix} \frac{1}{4}\mathcal{R}'\mathbf{Q}'(\tilde{\mathbf{q}})\mathbf{Q}(\tilde{\mathbf{q}})\mathcal{R}-(\mathbf{p})_\times^2 & -(\mathbf{p})_\times \\ (\mathbf{p})_\times & \mathbf{I} \end{bmatrix}$. For any $\mathbf{y} \in \mathbb{R}^3$,

$$\begin{aligned} \frac{1}{4}\mathbf{y}'\mathcal{R}'\mathbf{Q}'(\tilde{\mathbf{q}})\mathbf{Q}(\tilde{\mathbf{q}})\mathcal{R}\mathbf{y} &= \frac{1}{4}\left(\|\mathbf{y}\|^2 - (\mathbf{y}'\mathcal{R}'\tilde{\mathbf{q}}_q)^2\right) \\ &\geq \frac{\|\mathbf{y}\|^2}{4}\left(1-\|\tilde{\mathbf{q}}_q\|^2\right) \geq \|\mathbf{y}\|^2 c_{\mathcal{B}}, \end{aligned}$$

where $c_{\mathcal{B}} = \frac{1}{4}\left(1-\frac{c_{\max}}{8}\right)$. Therefore $\mathcal{B}(\tau,\lambda)\mathcal{B}'(\tau,\lambda) \geq \mathbf{B}(\tau)$, where $\mathbf{B}(\tau) := \begin{bmatrix} c_{\mathcal{B}}\mathbf{I}-(\mathbf{p})_\times^2 & -(\mathbf{p})_\times \\ (\mathbf{p})_\times & \mathbf{I} \end{bmatrix}$. Simple but long algebraic manipulations show that the eigenvalues of $\mathbf{B}(\tau)$ are given by $\alpha(\mathbf{B}(\tau)) \in \{\frac{1}{2}(1+c_{\mathcal{B}}+\|\mathbf{p}\| \pm \sqrt{(1+c_{\mathcal{B}}+\|\mathbf{p}\|)^2-4c_{\mathcal{B}}}), 1, c_{\mathcal{B}}\}$, which are positive and lower bounded by a positive constant c_B, independent of τ, if $\mathbf{p}$ is bounded, i.e. $\forall_\tau \alpha_{\min}(\mathbf{B}(\tau)) \geq c_B$ where $\alpha_{\min}(\mathbf{B}(\tau))$ denotes the smallest eigenvalue of $\mathbf{B}(\tau)$. Using the property $\mathbf{B}(\tau) \geq \alpha_{\min}(\mathbf{B}(\tau))\mathbf{I}$ produces $\mathcal{B}(\tau,\lambda)\mathcal{B}'(\tau,\lambda) \geq \alpha_{\min}(\mathbf{B}(\tau))\mathbf{I} \geq c_B\mathbf{I}$ and persistency of excitation condition is satisfied. Consequently, the parameterized LTV (36) is λ-UGES, and the nonlinear system (35) is exponentially stable in the domain $\mathcal{D}_q$. □

The exponential convergence derived in Theorem 11 is local in the sense that it is verified in the bounded region given by (34), i.e., given γ_p, γ_φ, γ_{b_ω}, and γ_{b_v}, any initial estimation error $\mathbf{x}_b(t_0)$ satisfying (34) converges exponentially fast to the origin. Since the convergence region is known, the observer design parameters can be used to guarantee exponential convergence for worst-case initial estimation errors. The following corollary establishes sufficient conditions such that the origin is uniformly exponentially stable for bounded initial estimation errors, which is a reasonable assumption for most applications.

Corollary 12. *Assume that the initial estimation errors are bounded*

$$\|\tilde{\mathbf{p}}(t_0)\| \leq \tilde{p}_0, \quad \|\mathbf{I}-\tilde{\mathcal{R}}(t_0)\|^2 \leq c_0 < 8, \tag{37a}$$

$$\|\tilde{\mathbf{b}}_\omega(t_0)\| \leq \tilde{b}_{\omega 0}, \quad \|\tilde{\mathbf{b}}_v(t_0)\| \leq \tilde{b}_{v0}, \tag{37b}$$

for some $\tilde{p}_0$, c_0, $\tilde{b}_{\omega0}$, $\tilde{b}_{v0}$, and let $(\gamma_p, \gamma_\varphi, \gamma_{b_\omega}, \gamma_{b_v})$ be such that $\gamma_{b_v}\tilde{b}_{v0}^2 + \gamma_p{}^B\tilde{p}_0^2 + \gamma_{b_\omega}\tilde{b}_{\omega0}^2 < \gamma_\varphi(8-c_0)$, and $\gamma_{b_\omega} = \gamma_{b_v}$ are satisfied. Then the equilibrium point $\mathbf{x}_b = (0,\mathbf{I},0,0)$ is uniformly exponentially stable in the set defined by (37).

Remark 4. The attitude inequality and the gain condition in Corollary 12 can be rewritten as $\varphi(t_0) \leq \varphi_0 < \pi$ and $\gamma_{b_v}\tilde{b}_{v0}^2 + \gamma_p{}^B\tilde{p}_0^2 + \gamma_{b_\omega}\tilde{b}_{\omega0}^2 < 4\gamma_\varphi(1+\cos(\varphi_0))$, respectively. The formulation in φ evidences that the stability property derived in Corollary 12 is independent of the rotation error axis $\boldsymbol{\phi}$. This enables the observer to operate on conditions where an upper bound $\varphi_0 < \pi$ for the initial estimation error is known, irrespective of the directionality of the attitude error.

Uniform stability guarantees upper bounds for the convergence rate in the set (37). Numerical convergence rate bounds can be computed by applying [29, Theorem 1 and Remark 2], however the obtained values are conservative. The conservativeness can be justified by the sufficiency of the adopted stability analysis tools based on parameterized LTVs, and by the fact that the computation of bounds for the matrix exponential is non-trivial in general [30].

The feedback laws can be expressed in terms of the landmark and velocity readings, (3) and (2) respectively.

Theorem 13. *The attitude and position feedback laws are explicit functions of the sensor readings and states estimates, and given by*

$$\hat{\omega} = \omega_r - \hat{\mathbf{b}}_\omega - k_\omega \mathbf{s}_\omega, \tag{38a}$$

$$\hat{\mathbf{v}} = \mathbf{v}_r - \hat{\mathbf{b}}_v + \left(\left(\omega_r - \hat{\mathbf{b}}_\omega\right)_\times - k_v \mathbf{I}\right)\mathbf{s}_v + k_\omega \left(\hat{\mathbf{p}}\right)_\times \mathbf{s}_\omega, \tag{38b}$$

$$\mathbf{s}_\omega = \sum_{i=1}^{n} (\hat{\mathcal{R}}'\mathbf{XDAe}_i) \times (\mathbf{QDAe}_i), \tag{38c}$$

$$\mathbf{s}_v = \hat{\mathbf{p}} + \frac{1}{n}\sum_{i=1}^{n} \mathbf{q}_i. \tag{38d}$$

Proof. The expressions (38a) and (38b) are directly obtained from (32). The feedback term $\mathbf{s}_\omega$ expressed in (31) is produced by taking (16) with the transformation $\mathbf{A}$ defined in Proposition 9. Consequently, $\mathbf{s}_\omega$ can be written in the form $\mathbf{s}_\omega = \mathcal{R}'\left(\mathbf{UU}'\tilde{\mathcal{R}} - \tilde{\mathcal{R}}'\mathbf{UU}'\right)_\otimes$, and following the proof of Theorem 6 produces $\mathbf{s}_\omega = \sum_{i=1}^{n} (\hat{\mathcal{R}}'\mathbf{XDAe}_i) \times (\mathbf{QDAe}_i)$, where $\mathbf{A}$ is defined such that $\mathbf{UU}' = \mathbf{I}$. The feedback term $\mathbf{s}_v$ is obtained from Theorem 6. □

5. Simulations

In this section, the proposed attitude and position observer properties are illustrated in simulation. A rigid body oscillating trajectory is considered, to analyze the stabilization of the position and attitude errors, the exponential convergence of the estimates, and the directionality brought about by the landmark configuration. Simulation results are presented for the cases of ideal and of biased velocity readings, studied in Section 3 and Section 4, respectively.

5.1. Ideal velocity readings

The landmarks are placed on the xy plane

$$^L\mathbf{x}_1 = \frac{1}{5}\begin{bmatrix} -4 \\ -3 \\ 0 \end{bmatrix}\text{ m},\ ^L\mathbf{x}_2 = \frac{1}{5}\begin{bmatrix} 2 \\ -3 \\ 0 \end{bmatrix}\text{ m},\ ^L\mathbf{x}_3 = \frac{1}{5}\begin{bmatrix} 2 \\ 6 \\ 0 \end{bmatrix}\text{ m}, \tag{39}$$

which satisfies the non-collinearity condition expressed in Assumption 1. The landmark coordinate transformation (9) is defined by $\mathbf{A} = \mathbf{I}$, and the matrix $\mathbf{P}$ defined in Proposition 1 and

Figure 2: Error of the position estimate with respect to local and to Earth frames (ideal velocity readings, $\varphi(t_0) = \frac{1}{3}\pi$ rad, $\boldsymbol{\phi}(t_0) = \frac{1}{\sqrt{3}}\mathbf{1}_3$).

(a)(b) NUoni- ufnoirfomrm IP, $\sigma_i(\mathbf{P}) \in$ {4.68, 3.24, 1.44}.

Figure 3: Attitude estimation error and exponential convergence bounds for diverse landmark coordinate transformations (ideal velocity readings, $\boldsymbol{\phi}(t_0) = \frac{1}{\sqrt{3}}\mathbf{1}'$).

its singular values and eigenvectors are given by

$$\mathbf{P} = \begin{bmatrix} 3.24 & 0 & 0 \\ 0 & 1.44 & 0 \\ 0 & 0 & 4.68 \end{bmatrix}, \quad \begin{matrix} \sigma_1(\mathbf{P}) = 4.68, & \mathbf{n}_1 = \begin{bmatrix} 0 & 0 & 1 \end{bmatrix}, \\ \sigma_2(\mathbf{P}) = 3.24, & \mathbf{n}_2 = \begin{bmatrix} 1 & 0 & 0 \end{bmatrix}, \\ \sigma_3(\mathbf{P}) = 1.44, & \mathbf{n}_3 = \begin{bmatrix} 0 & 1 & 0 \end{bmatrix}. \end{matrix}$$

The feedback gains are given by $k_v = k_\omega = 1$, and the rigid body trajectory is computed using oscillatory angular and linear rates of 1 Hz, and ideal velocity measurements.

The position estimation error $\tilde{\mathbf{p}}$ decreases exponentially fast to the origin, as illustrated in Fig. 2. The attitude error, shown in Fig. 3 for two different initial conditions, converges exponentially fast to the equilibrium point $\tilde{\mathcal{R}} = \mathbf{I}$, and is below the exponential bound (19). The convergence rate of the exponential bound is defined by the smallest singular value of $\mathbf{P}$, and provides for a worst-case convergence bound that is more conservative when $\sigma_1(\mathbf{P}) \gg \sigma_3(\mathbf{P})$, and tighter when the directionality of $\mathbf{P}$ is more uniform. This is evidenced in Fig. 3(b), where the convergence of the attitude error for a landmark transformation such that $\mathbf{P} = \mathbf{I}$ is shown. The actual convergence rate to the origin is slower for larger initial estimation error $\varphi(t_0)$, due to the stickiness effect [9] in the proximity of the anti-stable manifold defined by $\varphi = \pi$. A convincing discussion and illustration of the influence of anti-stable manifolds in the trajectories of nonlinear systems can be found in [8, 31].

The Euler axis trajectories in the hemisphere $\mathbf{n}_3'\boldsymbol{\phi} \geq 0$, de-

Figure 4: Euler axis trajectories on S(2).

picted in Fig. 4, illustrate the directionality of the attitude error discussed in Section 3.5. As derived in Theorem 7, the trajectories of the Euler axis converge to the direction associated with the smallest singular value of $\mathbf{P}$, that is $\boldsymbol{\phi}(t) \to \mathbf{n}_3$ as $t \to \infty$ for $\mathbf{n}_3'\boldsymbol{\phi} > 0$. Fig. 4 also shows that the boundary $\mathbf{n}_3'\boldsymbol{\phi} = 0$ is an invariant set of measure zero, and that the trajectories near $\mathbf{n}_3'\boldsymbol{\phi} = 0$ converge slower to $\mathbf{n}_3$, due to the stickiness effect of the set defined by $\mathbf{n}_3'\boldsymbol{\phi} = 0$.

5.2. Biased velocity readings

The attitude and position observer with biased velocity readings is analyzed using the planar landmark configuration (39). The landmark coordinate transformation $\mathbf{A}$ is designed so that $\mathbf{UU}' = \mathbf{I}$, using the constructive method presented in the proof of Proposition 9 and in Appendix B.2.

The values of γ_p, γ_φ and γ_b are computed to satisfy the condition of Corollary 12 for large bounds on the initial estimation errors, given by

$$\tilde{p}_0 = 2\sqrt{3}\,\text{m},\ \varphi_0 = \frac{\pi}{2}\,\text{rad},$$
$$\tilde{b}_{\omega 0} = 5\frac{\sqrt{3}\pi}{180}\,\text{rad/s},\ \tilde{b}_{v0} = \sqrt{3}\times 10^{-1}\,\text{m/s}. \quad (40)$$

The adopted values are given by $\gamma_\varphi = 1$, $\gamma_p = \frac{1}{4}$, and multiple values of γ_b are used to study the convergence of the observer, namely $\gamma_b \in \{0.19,\ 1.89\}$ that bear $(\frac{\gamma_\varphi}{\gamma_b}, \frac{\gamma_p}{\gamma_b}) \in \{(0.53, 0.13),\ (5.29, 1.32)\}$.

The initial attitude and position of the rigid body are $\mathcal{R} = \mathbf{I}$, $\mathbf{p} = \begin{bmatrix}1 & 1 & 1\end{bmatrix}'$ m, and the initial estimation errors are given by

$$\tilde{\mathbf{p}}(t_0) = \begin{bmatrix}-2\\2\\2\end{bmatrix}\text{m},\quad \varphi(t_0) = \frac{72\pi}{180}\,\text{rad},\quad \boldsymbol{\phi}(t_0) = \frac{1}{\sqrt{3}}\begin{bmatrix}1\\1\\1\end{bmatrix},$$
$$\tilde{\mathbf{b}}_\omega(t_0) = \frac{5\pi}{180}\begin{bmatrix}1\\-1\\1\end{bmatrix}\text{rad/s},\quad \tilde{\mathbf{b}}_v(t_0) = 10^{-1}\begin{bmatrix}1\\1\\-1\end{bmatrix}\text{m/s},$$

(a) Attitude. (b) Position.

Figure 5: Attitude and position estimation errors (biased velocity readings).

(a) Angular velocity bias. (b) Linear velocity bias.

Figure 6: Bias estimation error.

that are within the bounds (40). The rigid body trajectory is computed using oscillatory angular and linear velocities of 1 Hz.

The results for the case where both angular and linear velocity readings are biased, are presented in Figs. 5 and 6 . The convergence of the estimation error to the origin is faster for larger feedback gains. As expected, the compensation of bias is obtained at the cost of slower convergence of the attitude and position estimates, as evidenced by comparing Figs. 2 and 3 with Figs. 5(a) and 5(b) .

Larger gains introduce faster convergence, yet higher peaks in the bias estimates are also obtained. These can be justified by analyzing the level sets of the Lyapunov function $V_b \leq c$, that are positively invariant and contain points with small attitude and position error $\|\mathbf{I} - \tilde{\mathcal{R}}\| \approx 0$, $\|\tilde{\mathbf{p}}\| \approx 0$, but with large bias error $\|\tilde{\mathbf{b}}_\omega\|^2 + \|\tilde{\mathbf{b}}_v\|^2 \approx \frac{2c}{\gamma_b}$.

The Lyapunov function convergence is shown in Fig. 7, where the logarithmic scale is adopted to demonstrate exponentially fast convergence to the origin. Given that V_b provides for an upper bound for the estimation error $(\tilde{\mathbf{p}}, \tilde{\mathcal{R}}, \tilde{\mathbf{b}}_\omega, \tilde{\mathbf{b}}_v)$, Fig. 7 shows that, in spite of the peak values attained for $(\tilde{\mathbf{b}}_\omega,\ \tilde{\mathbf{b}}_v)$,

Figure 7: Exponential convergence of V_b (biased velocity readings).

the norm of the estimation error converges exponentially fast to $(\mathbf{0}, \mathbf{I}, \mathbf{0}, \mathbf{0})$.

6. Conclusions

A nonlinear observer for position and attitude estimation on SE(3) was proposed, using landmark measurements and non-ideal velocity readings. A Lyapunov function, conveniently defined by the landmark measurement error, was adopted to derive the position and attitude feedback laws. This approach provided for an insight on the necessary and sufficient landmark configuration for position and attitude estimation, and produced an output feedback architecture, expressed as a function of the sensor readings and state estimates.

The case of ideal velocity readings allowed for the decoupling of the position and attitude systems, and almost global asymptotic stability of the origin together with exponential convergence of the trajectories in any closed ball inside the region of attraction were obtained. The asymptotic behavior of the trajectories was also characterized, showing that the attitude error converges to the axis of the smallest eigenvalue of a matrix defined by the landmark geometry. The stability results were extended for the case of biased linear velocity readings, where the position and attitude systems were coupled by the presence of rate gyro bias. Using recently results for parameterized LTVs, exponential stabilization of the origin for bounded initial estimation errors was shown.

Simulation results illustrated the convergence properties of the observer for diverse feedback gains and initial conditions. The theoretical exponential convergence bounds were shown to be close to the real estimation error. Trajectories emanating from initial conditions near the anti-stable manifolds showed smaller convergence rate, as expected from the continuity of the solutions of dynamical systems. In the case of biased linear and angular velocity readings, exponential convergence to the origin, for initial conditions in a bounded region, was shown. The effects of time-varying velocities in the solutions of the nonautonomous error dynamics was negligible. The trade-off between convergence rate and the peak values of the estimates was justified using the level sets of the Lyapunov function. Future work will focus on improving the convergence rate bounds in the presence of biased velocity measurements, and on the exact discrete time implementation of the algorithm.

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A. Auxiliary results

This section contains elementary results from linear algebra that are adopted in this work.

Let $\mathbf{A}, \mathbf{B} \in \mathrm{M}(n, m)$ and denote the matrix columns by $\mathbf{a}_i, \mathbf{b}_i \in \mathbb{R}^n$, respectively. Let $\mathcal{R} \in \mathrm{SO}(n)$, $\mathbf{K} \in \mathrm{so}(n)$, $\mathbf{K}_3 \in \mathrm{so}(3)$, $\mathbf{S} \in \mathrm{L}(n)$ and $\{\mathbf{u}, \mathbf{v}\} \in \mathbb{R}^3$, then

$$\sum_{i=1}^{n} \mathbf{a}_i'\mathbf{b}_i = \mathrm{tr}(\mathbf{A}\mathbf{B}'),\ \sum_{i=1}^{n} \mathbf{a}_i\mathbf{b}_i' = \mathbf{A}\mathbf{B}',\ \mathrm{tr}(\mathbf{A}\mathbf{B}') = \mathrm{tr}(\mathbf{B}'\mathbf{A}),$$

$$\mathrm{tr}(\mathbf{K}\mathbf{S}) = 0,\ \mathrm{tr}(\mathbf{K}\mathbf{A}) = \mathrm{tr}\left(\mathbf{K}\frac{\mathbf{A}-\mathbf{A}'}{2}\right),\ \mathcal{R}(\mathbf{K}_3)_\otimes = (\mathcal{R}\mathbf{K}_3\mathcal{R}')_\otimes,$$

$$\mathbf{u}'\mathbf{v} = -\frac{1}{2}\mathrm{tr}((\mathbf{u})_\times(\mathbf{v})_\times),\ (\mathbf{u})_\times(\mathbf{v})_\times = \mathbf{v}\mathbf{u}' - \mathbf{v}'\mathbf{u}\mathbf{I}.$$

Lemma 14. *Let* $\mathbf{S} \in \mathrm{L}(3)$, $\mathcal{R} = \mathrm{rot}(\varphi, \boldsymbol{\phi}) \in \mathrm{SO}(3)$, *then*

$$(\mathbf{S}\mathcal{R} - \mathcal{R}'\mathbf{S})_\otimes = \mathbf{Q}'(\varphi, \boldsymbol{\phi})(\mathrm{tr}(\mathbf{S})\mathbf{I} - \mathbf{S})\boldsymbol{\phi},$$

where $\mathbf{Q}(\varphi, \boldsymbol{\phi}) = \sin(\varphi)\mathbf{I} + (1 - \cos(\varphi))(\boldsymbol{\phi})_\times$.

Proof. Using $\mathcal{R} = \mathbf{I} + \mathbf{Q}(\varphi, \boldsymbol{\phi})(\boldsymbol{\phi})_\times$, $\boldsymbol{\phi}'\mathbf{Q}(\varphi, \boldsymbol{\phi}) = \boldsymbol{\phi}'\sin(\varphi)$, and the properties of the trace, bears that, for any $\mathbf{a} \in \mathbb{R}^3$,

$$\begin{aligned}
&(\mathbf{S}\mathcal{R} - \mathcal{R}'\mathbf{S}')_\otimes'\mathbf{a} = -\mathrm{tr}((\mathbf{a})_\times\mathbf{S}\mathcal{R})\\
&= -\mathrm{tr}((\mathbf{a})_\times\mathbf{S}\mathbf{Q}(\varphi, \boldsymbol{\phi})(\boldsymbol{\phi})_\times) = -\mathrm{tr}((\mathbf{a}\boldsymbol{\phi}' - \boldsymbol{\phi}'\mathbf{a}\mathbf{I})\mathbf{S}\mathbf{Q}(\varphi, \boldsymbol{\phi}))\\
&= -\boldsymbol{\phi}'\mathbf{S}\mathbf{Q}(\varphi, \boldsymbol{\phi})\mathbf{a} + \mathrm{tr}(\mathbf{S})\boldsymbol{\phi}'\sin(\varphi)\mathbf{a} = -\boldsymbol{\phi}'\mathbf{S}\mathbf{Q}(\varphi, \boldsymbol{\phi})\mathbf{a}\\
&+ \mathrm{tr}(\mathbf{S})\boldsymbol{\phi}'\mathbf{Q}(\varphi, \boldsymbol{\phi})\mathbf{a} = \boldsymbol{\phi}'(\mathrm{tr}(\mathbf{S})\mathbf{I} - \mathbf{S})\mathbf{Q}(\varphi, \boldsymbol{\phi})\mathbf{a}
\end{aligned}$$

that concludes the proof. □

B. Extensions of the landmark based nonlinear observer

This section discusses some specific configurations of the landmark based nonlinear observer. The problem of estimating directly the position of the rigid body with respect to Earth frame is discussed, and the formulation of the landmark transformation for three landmark measurements is described.

B.1. Position estimation in Earth coordinate frame

This section discusses how the observer formulation can be modified to estimate position with respect to the origin of a desired coordinate frame $\{E\}$. As illustrated in Fig. 1, the position with respect to $\{E\}$ is described by $\mathbf{p}_E = \mathbf{p} + \mathcal{R}'^E\mathbf{t}_L$, where $^E\mathbf{t}_L$ represents the coordinates of the origin of $\{L\}$ with respect to $\{E\}$, expressed in $\{E\}$, and frames $\{L\}$ and $\{E\}$ have the same orientation by definition, $\mathcal{R} = {}^E_B\mathbf{R} = {}^L_B\mathbf{R}$. Note that an estimate $\hat{\mathbf{p}}_E$ can be constructed using $\hat{\mathbf{p}}$ and $\hat{\mathcal{R}}$, bearing exponentially fast convergence of the error $\tilde{\mathbf{p}}_E$. However, $\tilde{\mathbf{p}}_E = 0$ does not verify the exponential stability property as defined in the literature [28].

The kinematics of $\hat{\mathbf{p}}_E$ and $\tilde{\mathbf{p}}_E$ are given by

$$\dot{\hat{\mathbf{p}}}_E = \hat{\mathbf{v}} - (\hat{\omega})_\times \hat{\mathbf{p}}_E, \tag{41a}$$

$$\dot{\tilde{\mathbf{p}}}_E = (\hat{\mathbf{v}} - \mathbf{v}) - (\omega)_\times \tilde{\mathbf{p}}_E + (\hat{\mathbf{p}}_E)_\times (\hat{\omega} - \omega), \tag{41b}$$

and the landmark coordinates measured in body frame can be written as a function of $\mathbf{p}_E$, producing

$$\mathbf{q}_i = \mathcal{R}'^E\mathbf{x}_{E\,i} - \mathbf{p}_E, \tag{42}$$

where $^E\mathbf{x}_{E\,i} = {}^L\mathbf{x}_i + {}^E\mathbf{t}_L$ are the coordinates of landmark i in frame $\{E\}$. The structure of the position kinematics (41) and of the landmark readings (42), is identical to the structure of the position kinematics (5), (7) and landmark readings (3) considered in the observer derivation. Using this similarity, an exponential stable position observer for $\mathbf{p}_E$ is obtained. The observer is derived for any frame $\{E\}$ that satisfies the following condition.

Assumption 2. *The landmark coordinates $^E\mathbf{x}_{E\,i}$ are linearly dependent, i.e. there exist scalars α_i, not all zero, such that* $\sum_{i=1}^n \alpha_i {}^E\mathbf{x}_{E\,i} = 0$.

Note that frame $\{L\}$ verifies Assumption 2 by construction. The adopted feedback law is obtained by rewriting (24b) as

$$\hat{\mathbf{v}} = \mathbf{v}_r + ((\omega_r)_\times - k_v\mathbf{I})\mathbf{s}_{vE} + k_\omega (\hat{\mathbf{p}})_\times \mathbf{s}_\omega, \quad \mathbf{s}_{vE} = \hat{\mathbf{p}}_E - \mathbf{Q}_r\mathbf{d}_\alpha,$$

where $\mathbf{d}_\alpha = \begin{bmatrix}\alpha_1 & \dots & \alpha_n\end{bmatrix}'$ is defined in Assumption 2 and can be seen as the generalization of $\mathbf{d}_p = -\frac{1}{n}\mathbf{1}_n$. Analytical manipulation yields that $\mathbf{p}_E = \mathbf{Q}_r\mathbf{d}_\alpha$ producing the closed loop position error kinematics $\dot{\tilde{\mathbf{p}}}_E = -k_v\tilde{\mathbf{p}}_E$. The origin $\tilde{\mathbf{p}}_E = 0$ is exponentially stable, as desired.

B.2. Landmark coordinate transformation with minimal set of landmarks

Assumption 1 establishes that $\operatorname{rank}(\mathbf{X}) \geq 2$, however $\operatorname{rank}(\mathbf{H}) = \operatorname{rank}(\mathbf{X})$ and the coordinate transformation described in Proposition 9 requires $\mathbf{H} = \mathbf{XD}$ to be full rank. This section shows that, in case $\operatorname{rank}(\mathbf{X}) = 2$, it is possible to augment matrix $\mathbf{H}$ to produce $\mathbf{H}_a$ such that $\operatorname{rank}(\mathbf{H}_a) = 3$. Taking two linearly independent columns of $\mathbf{H}$, $^L\mathbf{h}_i$ and $^L\mathbf{h}_j$, the augmented matrix is given by $\mathbf{H}_a = \begin{bmatrix}\mathbf{H} & {}^L\mathbf{h}_i \times {}^L\mathbf{h}_j\end{bmatrix}$, which is full rank. Defining $\mathbf{U}_{Xa} := \mathbf{H}_a\mathbf{A}_{Xa}$, by the steps of the proof of Proposition 9 there is $\mathbf{A}_{Xa} \in \mathrm{M}(n+1)$ nonsingular such that $\mathbf{U}_{Xa}\mathbf{U}'_{Xa} = \mathbf{I}$, as desired.

The cross product is commutable with rotation transformations, $(\mathcal{R}'{}^L\mathbf{h}_i) \times (\mathcal{R}'{}^L\mathbf{h}_j) = \mathcal{R}'({}^L\mathbf{h}_i \times {}^L\mathbf{h}_j)$, so the representation of the augmented matrices in body coordinates is simply given by $^B\mathbf{U}_{Xa} = \mathcal{R}'\mathbf{U}_{Xa}$, $^B\hat{\mathbf{U}}_{Xa} = \hat{\mathcal{R}}'\mathbf{U}_{Xa}$, $\mathbf{U}_{Xa} = \mathbf{H}_a\mathbf{A}_{Xa}$. Therefore, the modified observer is obtained by replacing $\mathbf{U}$ and $\mathbf{H}$ by $\mathbf{U}_{Xa}$ and $\mathbf{H}_a$, respectively, and the derived observer properties are obtained by simple change of variables.

C. Uniform exponential stability of parameterized time-varying systems

The following result from [22] establishes that if the parameterized nonlinear system is exponentially stable uniformly in λ, then uniform exponential stability (independent of the initial conditions) of the associated nonlinear system can be inferred. This result is presented here for the sake of clarity.

Lemma 15 (λ-UGES and UES [22]). *Consider*

i) *the nonautonomous system* $\dot{y} = f(t, y)$ *where* $f : \mathbb{R}_{\geq 0} \times \mathcal{D}_y \to \mathbb{R}^n$ *is piecewise continuous in t and locally Lipschitz in y uniformly in t, and* $\mathcal{D}_y \subset \mathbb{R}^n$ *is a domain that contains the origin,*

ii) *the parameterized nonautonomous system* $\dot{x} = f_\lambda(t, \lambda, x)$*, where* $f_\lambda : \mathbb{R}_{\geq 0} \times \mathcal{D}_p \times \mathbb{R}^n \to \mathbb{R}^n$ *is continuous, locally Lipschitz uniformly in t and* λ*,* $\mathcal{D}_p = \mathbb{R}_{\geq 0} \times \mathcal{D}_\lambda$ *and* $\mathcal{D}_\lambda \subset \mathbb{R}^n$ *is a closed not necessarily compact set.*

Let $\mathcal{D}_y \subset \mathcal{D}_\lambda$ *and assume that* $x(t) = 0$ *is* λ*-UGES, i.e. there exist* k_e *and* $\gamma_e > 0$ *such that, for all* $t \geq t_0$*,* $\lambda \in \mathcal{D}_p$ *and* $x_0 \in \mathbb{R}^n$*, the solution of the system verifies* $\|x(t, \lambda, t_0, x_0)\| \leq k_e \|x_0\| e^{-\gamma_e(t-t_0)}$*. If the solution of both systems coincide,* $y(t, y_0, t_0) = x(t, \lambda, x_0, t_0)$*, for* $\lambda = (t_0, y_0)$ *and* $x_0 = y_0$*, then* $y(t) = 0$ *is exponentially stable in* $\mathcal{D}_y$*.*

Proof. Let $x_0 = y_0$ and $\lambda = (t_0, y_0)$, then $x(t, \lambda, t_0, x_0) = y(t, t_0, y_0)$ and by change of variables, the solution satisfies $\|y(t, t_0, y_0)\| \leq k_e \|y_0\| e^{-\gamma_e(t-t_0)}$, and uniform exponential stability of $y(t) = 0$ in $\mathcal{D}_y$ is immediate. □

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