On the Observability of Linear Motion Quantities in Navigation Systems

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Abstract

Navigation systems are a key element in a large variety of mobile platforms, where the correct knowledge of their position and attitude is essential in most applications. This paper focuses on the observability of linear motion quantities (position, linear velocity, linear acceleration, and accelerometer bias). It presents necessary and sufficient conditions, with clear physical insight, for the observability of these variables in 3-D. The analysis provided is based on kinematic models, which are exact and intrinsic to the motion of a rigid-body, and different cases are presented depending on the assumptions made on the sensor suite that is available on-board.

Key words: Linear time-varying systems; observability; 3-D linear motion kinematics; navigation systems.

1. Introduction

The design of Integrated Navigation Systems arises naturally in the development of a large variety of mobile platforms, whether manned or unmanned, autonomous or human-operated, as the knowledge of the position and attitude of the vehicle is a basic requirement for its successful operation. Moreover, for control purposes, other quantities such as the linear and angular velocities are also often required.

Dead-reckoning navigation systems such as Inertial Navigation Systems (INS) provide all these quantities. However, the estimation of the position and attitude of the vehicle is necessarily obtained in this type of systems by integrating higher-order derivatives such as the linear acceleration and the angular velocity, which are measured using, e.g., an Inertial Measurement Unit (IMU). As such, and regardless of the accuracy and precision of the IMU, the errors in the position and attitude estimates grow unbounded due to non-idealities such as noise and bias that affect the IMU readings [1]. These intrinsic limitations of dead-reckoning navigation systems are usually tackled by using aiding devices such as position and attitude sensors, e.g., the popular Global Positioning System (GPS), inclinometers, and magnetometers. However, even with the inclusion of aiding devices, not all states are always observable, in particular, if biases are considered and the acceleration of gravity is not known with enough accuracy. This paper investigates the observability of linear motion quantities of mobile platforms.

Previous work on the study of observability of navigation systems can be found in the literature. In [2] the observability of INS during initial alignment and calibration at rest is analyzed. The nominal nonlinear navigation system dynamics are perturbed yielding linearized error dynamics which are then shown not to be completely observable. In [3] the observability of a linearized INS error model is also examined for a stationary vehicle and it is reported, among other results concerning the leveling errors, that the unobservable states, which are distributed in two decoupled subspaces, can be systematically determined. In-flight alignment of INS is studied in [4] where it is shown that its observability can be improved by adequately maneuvering. In [5] sufficient conditions for the observability of stationary Strapdown Inertial Navigation Systems (SDINS) are analytically derived. In [6] an observability analysis of a GPS/INS system during two types of maneuvers, linear acceleration and steady turn, is presented. The analysis is based on a perturbation model of the INS and it is shown that the observability is improved when the vehicle maneuvers. Observability properties of the errors in an integrated navigation system are studied in [7], where the authors show that acceleration changes improve the estimates of attitude and rate-gyro bias and changes of the angular velocity enhance the lever arm estimate. However, no theoretical results for non-trivial trajectories are given and only simulation results are provided, which confirm that the degree of observability of the system increases with the richness of the trajectories described by the vehicle. To the best of the authors knowledge, in the literature only local observability results are known, most of them obtained in the context of navigation systems designed around the Extended Kalman Filter (EKF). These results, that reflect the continued adoption of EKF techniques to solve the Navigation problem, are very intuitive and were fundamental to motivate the need for the analysis presented in this paper. A related study on the observability of perspective systems can be found in [8], which has application to vision-based systems with perspective outputs. In [9] the authors propose a locally convergent observer for the attitude, in 3-D, using line-based dynamic vision, and also discuss the ob-

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servability of the corresponding system, revealing interesting group properties tied to the underlying system structure.

This paper presents a detailed analytical study on the maneuvers that can improve observability and provides necessary and sufficient conditions for the observability of linear motion quantities (position, linear velocity, linear acceleration, and accelerometer bias) assuming exact angular measurements. Four different sensor suites are considered and definite results are provided for all of them. The analysis is based on kinematic models, which are exact and intrinsic to the motion of the vehicle, and builds on well established observability results for linear time-varying (LTV) and linear time invariant (LTI) systems. For LTI systems, the concept of observability suffices to synthesize a globally asymptotically stable observer or filter. For LTV systems, stronger forms of observability should be considered. As such, the present work provides not only observability conditions but also results regarding uniform complete observability, which allow to derive globally asymptotically stable observers or filtering solutions, see [10]. Preliminary work by the authors can be found in [11].

The paper is organized as follows. In Section 2 some basic observability definitions and results are briefly presented for the sake of completeness. The linear motion dynamic systems whose observability is studied are introduced in Section 3, while the main results of the paper are derived and discussed in Section 4. Section 5 summarizes the main conclusions of the paper.

1.1. Notation

Throughout the paper the symbol $\mathbf{0}$ denotes a matrix (or vector) of zeros and $\mathbf{I}$ an identity matrix, both of appropriate dimensions. A block diagonal matrix is represented as $\text{diag}(\mathbf{A}_1, \ldots, \mathbf{A}_n)$ and, if $\mathbf{X}$ is a complex-valued matrix, $\mathbf{X}^T$ and $\mathbf{X}^*$ denote its transpose and conjugate transpose, respectively. For $\mathbf{x}, \mathbf{y} \in \mathbb{R}^3$, $\mathbf{x} \times \mathbf{y}$ represents the cross product. The pure unit imaginary number is defined as $j := \sqrt{-1}$ and the Special Orthogonal Group is denoted by $SO(3)$.

2. Preliminary Observability Definitions

Consider the LTV system

$$\begin{cases} \dot{\mathbf{x}}(t) = \mathbf{A}(t)\mathbf{x}(t) + \mathbf{B}(t)\mathbf{u}(t) \\ \mathbf{y}(t) = \mathbf{C}(t)\mathbf{x}(t) \end{cases}, \tag{1}$$

where $\mathbf{x}$, $\mathbf{u}$, and $\mathbf{y}$ are the state, input, and output of the system, respectively, $t \in [t_0, +\infty[$, and $\mathbf{A}(t)$, $\mathbf{B}(t)$, and $\mathbf{C}(t)$ are continuous matrices of compatible dimensions.

Definition 2.1 (Observability)**.** The LTV system (1) is called *observable on* $\left[t_0, t_f\right]$ if any initial state $\mathbf{x}(t_0)$ is uniquely determined by the corresponding output $\left\{\mathbf{y}(t),\ t \in \left[t_0, t_f\right]\right\}$.

Definition 2.2 (Observability Gramian and Transition Matrix)**.** The observability Gramian associated with the pair $(\mathbf{A}(t), \mathbf{C}(t))$, denoted as $\mathcal{W}\left(t_0, t_f\right)$, is given by

$$\mathcal{W}\left(t_0, t_f\right) = \int_{t_0}^{t_f} \boldsymbol{\phi}^T(t,\ t_0)\, \mathbf{C}^T(t)\mathbf{C}(t)\boldsymbol{\phi}(t,\ t_0)\, dt,$$

where

$$\boldsymbol{\phi}(t,\ t_0) = \mathbf{I} + \int_{t_0}^{t} \mathbf{A}(\sigma_1)\, d\sigma_1 + \int_{t_0}^{t} \mathbf{A}(\sigma_1) \int_{t_0}^{\sigma_1} \mathbf{A}(\sigma_2)\, d\sigma_2 d\sigma_1 + \ldots$$

is the transition matrix associated with $\mathbf{A}(t)$.

Theorem 2.1. *The LTV system* (1) *is observable on* $\left[t_0, t_f\right]$ *if and only if* $\mathcal{W}\left(t_0, t_f\right)$ *is invertible.*

Definition 2.3 (Uniform complete observability)**.** The LTV system (1) is called *uniformly completely observable* if there exist positive constants δ, α_1, and α_2 such that

$$\alpha_1 \mathbf{I} \leq \mathcal{W}(t, t+\delta) \leq \alpha_2 \mathbf{I} \tag{2}$$

for all $t \geq t_0$.

Remark 1. When the system matrices $\mathbf{A}(t)$ and $\mathbf{C}(t)$ are norm-bounded, it is easy to see that the right side of (2) is always satisfied. This is the case of the systems under study in the paper and therefore only the left side of (2) is considered and the existence of α_2 needs not to be addressed.

Remark 2. It is important to refer that Definition 2.3 applies only to bounded realizations, which are in fact those considered in the paper. For a more detailed discussion on the concept of uniform complete observability, the reader is referred to [12], while alternative criteria for uniform complete controllability/observability can be found in [13] and [14].

3. Linear Motion Kinematics

Let $\{I\}$ be an inertial coordinate frame and $\{B\}$ the body-fixed coordinate frame, whose origin coincides with the center of mass of the vehicle. Let ${}^I\mathbf{p}(t) \in \mathbb{R}^3$ denote the position of the origin of $\{B\}$, described in $\{I\}$, and $\mathbf{v}(t) \in \mathbb{R}^3$ the velocity of the vehicle relative to $\{I\}$, expressed in body-fixed coordinates. The linear motion kinematics of the vehicle are given by

$$\frac{d}{dt}{}^I\mathbf{p}(t) = \mathbf{R}(t)\mathbf{v}(t), \tag{3}$$

where $\mathbf{R}(t) \in SO(3)$ is the rotation matrix from body-fixed to inertial coordinates, i.e., from $\{B\}$ to $\{I\}$, that satisfies

$$\dot{\mathbf{R}}(t) = \mathbf{R}(t)\mathbf{S}\left[\boldsymbol{\omega}(t)\right],$$

where $\boldsymbol{\omega}(t) \in \mathbb{R}^3$ is the angular velocity of the vehicle, expressed in body-fixed coordinates, and $\mathbf{S}(\boldsymbol{\omega})\, \mathbb{R}^{3\times 3}$ is the skew-symmetric matrix such that $\mathbf{S}(\boldsymbol{\omega})\, \boldsymbol{x}$ is the cross product $\boldsymbol{\omega} \times \boldsymbol{x}$. The position of the vehicle in inertial coordinates is often available, e.g., when there is a GPS receiver installed on-board. However, in underwater robotics, for instance, GPS is unavailable and alternative positioning sensors are required [15]. Acoustic positioning systems are common, e.g., long baseline (LBL) or ultra-short baseline (USBL) sensors. In the latter case, the USBL (in the so-called inverted configuration) typically measures the position of an external fixed mark relative to the position of the vehicle, expressed in body-fixed coordinates, and thus the position of the vehicle is only available indirectly. Indeed, if $\mathbf{p}(t) \in \mathbb{R}^3$ denotes the measurement of the USBL as it was just described, it satisfies

$$\mathbf{p}(t) = \mathbf{R}^T(t)\left[{}^I\mathbf{p}_m - {}^I\mathbf{p}(t)\right],$$

where ${}^I\mathbf{p}_m \in \mathbb{R}^3$ denotes the inertial position of the mark. In this framework, the kinematics of the vehicle can be described, indirectly, by

$$\dot{\mathbf{p}}(t) = -\mathbf{S}\left[\boldsymbol{\omega}(t)\right]\mathbf{p}(t) - \mathbf{v}(t). \tag{4}$$

An essential element of Navigation Systems is the IMU, which usually contains two triads of orthogonally mounted accelerometers and rate gyros. Assuming that the IMU is installed at the center of mass of the vehicle and aligned with the body-fixed coordinate frame $\{B\}$, the rate gyros provide the angular velocity of the vehicle, $\boldsymbol{\omega}(t)$, and the accelerometers measure

$$\mathbf{a}(t) = \dot{\mathbf{v}}(t) + \mathbf{S}\left[\boldsymbol{\omega}(t)\right]\mathbf{v}(t) - \mathbf{g}(t) + \mathbf{b}(t), \tag{5}$$

where $\mathbf{g}(t) \in \mathbb{R}^3$ denotes the acceleration of gravity and $\mathbf{b}(t) \in \mathbb{R}^3$ the bias of the accelerometer, both expressed in body-fixed coordinates. Ideal accelerometers would not measure the gravitational term but in practice this term must be considered due to the inherent physics of these sensors, see [16] for further details. The term $\mathbf{S}\left[\boldsymbol{\omega}(t)\right]\mathbf{v}(t)$ corresponds to the Coriolis acceleration of the vehicle and must also be considered. The measurements provided by the rate gyros are also usually corrupted by biases. However, these biases can be compensated using an Attitude and Heading Reference System (AHRS) and there are are several solutions for this problem in the literature, see e.g. [17] and [18] for almost globally asymptotically stable attitude observers, which also account for the rate gyro bias, or [19] for a globally asymptotically stable solution.

In the remainder of this section four different dynamic systems will be introduced that describe the linear motion of the vehicle and its relation with the various sensors. The differences between the proposed dynamics depend upon the sensor suite considered. As it was seen, both (3) and (4) describe the evolution of the position of the vehicle given the information provided by the sensors installed on-board. In what concerns observability properties they are equivalent assuming exact angular measurements. Throughout the paper, and without loss of generality, (4) is preferred due to its particular structure. Finally, it is assumed that $\boldsymbol{\omega}(t)$ and its derivative are bounded, and that $\boldsymbol{\omega}(t)$ is continuous, which is true for all manned and unmanned platforms.

3.1. Navigation with calibrated accelerometer

The first case considered in the paper is not, at first, a simple one, but its observability analysis turns out to be quite straightforward after an appropriate state transformation. It is considered here that the vehicle is equipped with a positioning sensor and a calibrated accelerometer, together with a triad of rate gyros or an AHRS, to provide the angular velocity of the vehicle.

The derivative of the linear position is given by (4), whereas the derivative of the velocity may be obtained from (5). The acceleration of gravity is assumed locally constant in inertial coordinates. Thus, the derivative of this quantity when expressed in body-fixed coordinates is given by

$$\dot{\mathbf{g}}(t) = -\mathbf{S}\left[\boldsymbol{\omega}(t)\right]\mathbf{g}(t).$$

The system dynamics can then be written as

$$\begin{cases} \dot{\mathbf{p}}(t) = -\mathbf{S}\left[\boldsymbol{\omega}(t)\right]\mathbf{p}(t) - \mathbf{v}(t) \\ \dot{\mathbf{v}}(t) = -\mathbf{S}\left[\boldsymbol{\omega}(t)\right]\mathbf{v}(t) + \mathbf{g}(t) + \mathbf{a}(t) \\ \dot{\mathbf{g}}(t) = -\mathbf{S}\left[\boldsymbol{\omega}(t)\right]\mathbf{g}(t) \\ \mathbf{y}(t) = \mathbf{p}(t) \end{cases}, \tag{6}$$

where $\mathbf{a}(t)$ is here considered as a deterministic input and $\mathbf{y}(t)$ denotes the system output, available for the estimation of the system state.

3.2. Navigation with known gravity

In Section 3.1 the gravity was unknown and the accelerometer was assumed to be calibrated. In this section the accelerometer measurements are assumed corrupted by an unknown bias but the gravity is supposed to be known. Although possible from the practical point of view, e.g., if the magnitude of the gravity is known, as well as the attitude of the vehicle, this is not a very useful situation as any misalignment in the gravity acceleration vector expressed in body-fixed coordinates may result in severe problems in the overall acceleration compensation. Nevertheless, it presents an interesting theoretical problem and provides insight to the more general setup, which is presented in Section 3.4. Moreover, it is also found in practical applications when a high-accuracy AHRS is available, which allows to determine the acceleration of gravity in body-fixed coordinates with enough accuracy. The system dynamics that reflect these assumptions are given by

$$\begin{cases} \dot{\mathbf{p}}(t) = -\mathbf{S}\left[\boldsymbol{\omega}(t)\right]\mathbf{p}(t) - \mathbf{v}(t) \\ \dot{\mathbf{v}}(t) = -\mathbf{S}\left[\boldsymbol{\omega}(t)\right]\mathbf{v}(t) - \mathbf{b}(t) + \mathbf{a}(t) + \mathbf{g}(t) \\ \dot{\mathbf{b}}(t) = \mathbf{0} \\ \mathbf{y}(t) = \mathbf{p}(t) \end{cases}, \tag{7}$$

where $\mathbf{a}(t)$ and $\mathbf{g}(t)$ are assumed to be deterministic inputs.

3.3. Dynamic accelerometer bias estimation

This section introduces a class of systems suitable for the estimation of the bias of an accelerometer assuming exact angular and linear velocity measurements, in body-fixed coordinates. This is particularly interesting, for example, if one has available a calibration table which permits the generation of high-resolution trajectories with known velocities. Furthermore, the most general setup, which is presented in Section 3.4, is an extension of this framework and therefore insight on the observability of this setup translates into insight on the observability properties of the most general setup. The system dynamics read as

$$\begin{cases} \dot{\mathbf{v}}(t) = -\mathbf{S}\left[\boldsymbol{\omega}(t)\right]\mathbf{v}(t) + \mathbf{g}(t) - \mathbf{b}(t) + \mathbf{a}(t) \\ \dot{\mathbf{g}}(t) = -\mathbf{S}\left[\boldsymbol{\omega}(t)\right]\mathbf{g}(t) \\ \dot{\mathbf{b}}(t) = \mathbf{0} \\ \mathbf{y}(t) = \mathbf{v}(t) \end{cases}, \tag{8}$$

where $\mathbf{a}(t)$ is again assumed to be a deterministic input and the output of the system is the velocity of the origin of the body-fixed coordinate frame.

3.4. Navigation with gravity and accelerometer bias dynamic estimation

The general setup regarding the estimation of linear motion quantities of mobile platforms is presented in this section. Both the acceleration of gravity and the bias of the accelerometer are supposed unknown and the system dynamics can be written as

$$\begin{cases} \dot{\mathbf{p}}(t) = -\mathbf{S}\left[\omega(t)\right]\mathbf{p}(t) - \mathbf{v}(t) \\ \dot{\mathbf{v}}(t) = -\mathbf{S}\left[\omega(t)\right]\mathbf{v}(t) + \mathbf{g}(t) - \mathbf{b}(t) + \mathbf{a}(t) \\ \dot{\mathbf{g}}(t) = -\mathbf{S}\left[\omega(t)\right]\mathbf{g}(t) \\ \dot{\mathbf{b}}(t) = \mathbf{0} \\ \mathbf{y}(t) = \mathbf{p}(t) \end{cases}, \qquad (9)$$

where $\mathbf{a}(t)$ is assumed to be a deterministic input.

4. Main Results

4.1. Navigation with calibrated accelerometer

This section examines the observability of the dynamic system (6), which has been derived in the past by the authors to propose a navigation filter with a calibrated accelerometer. In [20] it was shown that the system is observable. In practice, stronger forms of observability are convenient in order to guarantee the stability of state observers or filters. That is established in the following theorem.

Theorem 4.1. *The LTV system* (6) *is uniformly completely observable.*

Proof. In compact form, the dynamic system (6) can be rewritten as

$$\begin{cases} \dot{\mathbf{x}}_1(t) = \mathbf{A}_1(t)\mathbf{x}_1(t) + \mathbf{B}_1\mathbf{u}_1(t) \\ \mathbf{y}_1(t) = \mathbf{C}_1\mathbf{x}_1(t) \end{cases},$$

where $\mathbf{u}_1(t) = \mathbf{a}(t)$ is the input of the system,

$$\mathbf{x}_1(t) = \begin{bmatrix} \mathbf{p}(t) \\ \mathbf{v}(t) \\ \mathbf{g}(t) \end{bmatrix} \in \mathbb{R}^9$$

is the vector of states of the system,

$$\mathbf{A}_1(t) = \begin{bmatrix} -\mathbf{S}\left[\omega(t)\right] & -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{S}\left[\omega(t)\right] & \mathbf{I} \\ \mathbf{0} & \mathbf{0} & -\mathbf{S}\left[\omega(t)\right] \end{bmatrix},$$

$$\mathbf{B}_1 = \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \\ \mathbf{0} \end{bmatrix},$$

and $\mathbf{C}_1 = [\mathbf{I}\ \mathbf{0}\ \mathbf{0}]$. Consider the state transformation

$$\mathbf{x}_{\bar{1}}(t) := \mathbf{T}_1(t)\mathbf{x}_1(t),$$

with

$$\mathbf{T}_1(t) := \operatorname{diag}\left(\mathbf{R}(\mathrm{t}),\ \mathbf{R}(\mathrm{t}),\ \mathbf{R}(\mathrm{t})\right). \qquad (10)$$

Notice that (10) is a Lyapunov transformation matrix as

- $\mathbf{T}_1(t)$ is continuously differentiable for all t;
- Both $\mathbf{T}_1(t)$ and $\dot{\mathbf{T}}_1(t)$ are bounded for all t, where

$$\dot{\mathbf{T}}_1(t) = \operatorname{diag}\left(\mathbf{R}(\mathrm{t})\mathbf{S}\left[\omega(\mathrm{t})\right],\ \mathbf{R}(\mathrm{t})\mathbf{S}\left[\omega(\mathrm{t})\right],\ \mathbf{R}(\mathrm{t})\mathbf{S}\left[\omega(\mathrm{t})\right]\right);$$

- $\det\left[\mathbf{T}_1(t)\right] = 1$.

Then, the new system dynamics can be written as

$$\begin{cases} \dot{\mathbf{x}}_{\bar{1}}(t) = \mathbf{A}_{\bar{1}}\mathbf{x}_{\bar{1}}(t) + \mathbf{B}_{\bar{1}}(t)\mathbf{u}_1(t) \\ \mathbf{y}_{\bar{1}}(t) = \mathbf{C}_{\bar{1}}\mathbf{x}_{\bar{1}}(t) \end{cases}, \qquad (11)$$

where

$$\mathbf{A}_{\bar{1}} = \begin{bmatrix} \mathbf{0} & -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix},\ \mathbf{B}_{\bar{1}}(t) = \begin{bmatrix} \mathbf{0} \\ \mathbf{R}(t) \\ \mathbf{0} \end{bmatrix},$$

and $\mathbf{C}_{\bar{1}}(t) = \left[\mathbf{R}^T(t)\ \mathbf{0}\ \mathbf{0}\right]$. It is easy to compute the observability Gramian associated with the pair $\left(\mathbf{A}_{\bar{1}},\ \mathbf{C}_{\bar{1}}(t)\right)$ on $[t, t+\delta]$, given by

$$\mathcal{W}_{\bar{1}}(t, t+\delta) = \begin{bmatrix} \delta\mathbf{I} & -\frac{\delta^2}{2}\mathbf{I} & -\frac{\delta^3}{6}\mathbf{I} \\ -\frac{\delta^2}{2}\mathbf{I} & \frac{\delta^3}{3}\mathbf{I} & \frac{\delta^4}{8}\mathbf{I} \\ -\frac{\delta^3}{6}\mathbf{I} & \frac{\delta^4}{8}\mathbf{I} & \frac{\delta^5}{20}\mathbf{I} \end{bmatrix},$$

which does not depend on t and is positive definite for all $\delta > 0$. Moreover, for any fixed $\delta > 0$, there exists a lower bound for the minimum eigenvalue of $\mathcal{W}_{\bar{1}}(t, t+\delta)$. Therefore, (11) is uniformly completely observable, and it follows that (6) is also uniformly completely observable as both systems are related through a Lyapunov transformation [21]. □

4.2. Navigation with known gravity

This section examines the observability of the dynamic system (7). Notice that, for constant angular velocity the system is always observable. Thus, one can expect the system to be always observable, as in Section 4.1. Before going into the observability analysis, the following proposition is introduced.

Proposition 4.2. *Let* $\mathbf{f}(t) : \left[t_0, t_f\right] \subset \mathbb{R} \to \mathbb{R}^n$ *be a continuous and i-times continuously differentiable function on* $\mathcal{I} := \left[t_0, t_f\right]$, $T := t_f - t_0 > 0$*, and such that*

$$\mathbf{f}(t_0) = \dot{\mathbf{f}}(t_0) = \ldots = \mathbf{f}^{(i-1)}(t_0) = \mathbf{0}.$$

Further assume that

$$\max_{t \in \mathcal{I}} \left\|\mathbf{f}^{(i+1)}(t)\right\| \leq C.$$

If

$$\underset{\substack{\alpha > 0 \\ t_1 \in \mathcal{I}}}{\exists} \ : \ \left\|\mathbf{f}^{(i)}(t_1)\right\| \geq \alpha,$$

then

$$\underset{\substack{0 < \delta \leq T \\ \beta > 0}}{\exists} \ : \ \left\|\mathbf{f}(t_0 + \delta)\right\| \geq \beta.$$

Proof. The proof is presented in Appendix A. □

The following theorem is the main result of this section.

Theorem 4.3. *The LTV system* (7) *is uniformly completely observable.*

Proof. In compact form, the dynamic system (7) can be rewritten as

$$\begin{cases} \dot{\mathbf{x}}_2(t) = \mathbf{A}_2(t)\mathbf{x}_2(t) + \mathbf{B}_2\mathbf{u}_2(t) \\ \mathbf{y}_2(t) = \mathbf{C}_2\mathbf{x}_2(t) \end{cases}, \tag{12}$$

where

$$\mathbf{u}_2(t) = \begin{bmatrix} \mathbf{a}(t) \\ \mathbf{g}(t) \end{bmatrix} \in \mathbb{R}^6$$

is the input of the system,

$$\mathbf{x}_2(t) = \begin{bmatrix} \mathbf{p}(t) \\ \mathbf{v}(t) \\ \mathbf{b}(t) \end{bmatrix} \in \mathbb{R}^9$$

is the vector of states of the system,

$$\mathbf{A}_2(t) = \begin{bmatrix} -\mathbf{S}[\omega(t)] & -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & -\mathbf{S}[\omega(t)] & -\mathbf{I} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix},$$

$$\mathbf{B}_2 = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{I} & \mathbf{I} \\ \mathbf{0} & \mathbf{0} \end{bmatrix},$$

and $\mathbf{C}_2 = [\mathbf{I}\ \mathbf{0}\ \mathbf{0}]$. Consider the Lyapunov transformation

$$\mathbf{x}_{\bar{2}}(t) := \mathbf{T}_2(t)\mathbf{x}_2(t),$$

with

$$\mathbf{T}_2(t) := \operatorname{diag}(\mathbf{R}(\mathrm{t}), \mathbf{R}(\mathrm{t}), \mathbf{I}).$$

Then, the new system dynamics can be written as

$$\begin{cases} \dot{\mathbf{x}}_{\bar{2}}(t) = \mathbf{A}_{\bar{2}}(t)\mathbf{x}_{\bar{2}}(t) + \mathbf{B}_{\bar{2}}(t)\mathbf{u}_2(t) \\ \mathbf{y}_{\bar{2}}(t) = \mathbf{C}_{\bar{2}}(t)\mathbf{x}_{\bar{2}}(t) \end{cases},$$

where

$$\mathbf{A}_{\bar{2}}(t) = \begin{bmatrix} \mathbf{0} & -\mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{R}(t) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix},$$

$$\mathbf{B}_{\bar{2}}(t) = \begin{bmatrix} \mathbf{0} & \mathbf{0} \\ \mathbf{R}(t) & \mathbf{R}(t) \\ \mathbf{0} & \mathbf{0} \end{bmatrix},$$

and $\mathbf{C}_{\bar{2}}(t) = \left[\mathbf{R}^T(t)\ \mathbf{0}\ \mathbf{0}\right]$. Let

$$\mathbf{R}^{[1]}(t, t_0) := \int_{t_0}^{t} \mathbf{R}(\sigma)\, d\sigma$$

and

$$\mathbf{R}^{[i]}(t, t_0) := \int_{t_0}^{t} \dots \int_{t_0}^{\sigma_{i-1}} \mathbf{R}(\sigma_i)\, d\sigma_i \dots d\sigma_1,$$

where $(.)^{[i]}$ represents the $i-th$ integral of the quantity. Then, it is a simple matter of computation to show that the transition matrix associated with $\mathbf{A}_{\bar{2}}(t)$ is given by

$$\boldsymbol{\phi}_{\bar{2}}(t, t_0) = \begin{bmatrix} \mathbf{I} & -(t-t_0)\mathbf{I} & \mathbf{R}^{[2]}(t, t_0) \\ \mathbf{0} & \mathbf{I} & -\mathbf{R}^{[1]}(t, t_0) \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}$$

and, if $\mathcal{W}_{\bar{2}}\left(t_0, t_f\right)$ denotes the observability Gramian associated with the pair $\left(\mathbf{A}_{\bar{2}}(t), \mathbf{C}_{\bar{2}}(t)\right)$,

$$\mathbf{d}^T \mathcal{W}_{\bar{2}}\left(t_0, t_f\right) \mathbf{d} = \int_{t_0}^{t_f} \left\| \mathbf{d}_1 - (\tau - t_0)\, \mathbf{d}_2 + \mathbf{R}^{[2]}(\tau, t_0)\, \mathbf{d}_3 \right\|^2 d\tau$$

for all

$$\mathbf{d} = \begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \\ \mathbf{d}_3 \end{bmatrix} \in \mathbb{R}^9,\ \|\mathbf{d}\| = 1.$$

Define

$$\mathbf{f}(\tau) := \mathbf{d}_1 - (\tau - t)\, \mathbf{d}_2 + \mathbf{R}^{[2]}(\tau, t)\, \mathbf{d}_3$$

for all $t \geq t_0$, $\delta > 0$, and $\tau \in [t, t+\delta]$. Notice that

$$\mathbf{d}^T \mathcal{W}_{\bar{2}}(t, t+\delta)\, \mathbf{d} = \int_{t}^{t+\delta} \|\mathbf{f}(\tau)\|^2 d\tau.$$

The first three derivatives of $\mathbf{f}(\tau)$ are given by

$$\frac{d}{d\tau}\mathbf{f}(\tau) = -\mathbf{d}_2 + \mathbf{R}^{[1]}(\tau, t)\, \mathbf{d}_3,$$

$$\frac{d^2}{d\tau^2}\mathbf{f}(\tau) = \mathbf{R}(\tau)\, \mathbf{d}_3,$$

and

$$\frac{d^3}{d\tau^3}\mathbf{f}(\tau) = \mathbf{R}(\tau)\, \mathbf{S}[\omega(\tau)]\, \mathbf{d}_3.$$

Notice that all three derivatives are norm bounded for $\tau \in [t, t+\delta]$, uniformly in t. Suppose that $\mathbf{d}_1 \neq \mathbf{0}$. Then, there exists $\alpha_1 > 0$ such that

$$\|\mathbf{f}(t)\|^2 = \alpha_1^2$$

for all $t \geq t_0$. Moreover, notice that $\frac{d}{d\tau}\|\mathbf{f}(\tau)\|^2$ has an upper bound, which does not depend on t. As, in addition to that,

$$\mathbf{d}^T \mathcal{W}_{\bar{2}}(t, t)\, \mathbf{d} = \int_{t}^{t} \|\mathbf{f}(\tau)\|^2 d\tau = 0$$

for all $t \geq t_0$ it follows, using Proposition 4.2, that

$$\underset{\substack{\delta_1 > 0 \\ \beta_1 > 0}}{\exists} \quad \underset{t \geq t_0}{\forall} \quad : \quad \mathbf{d}^T \mathcal{W}_{\bar{2}}(t, t+\delta_1)\, \mathbf{d} \geq \beta_1.$$

Suppose now that $\mathbf{d}_1 = \mathbf{0}$ and $\mathbf{d}_2 \neq \mathbf{0}$. Then, there exists $\alpha_2 > 0$ such that

$$\left\| \frac{d}{d\tau}\mathbf{f}(\tau)\bigg|_{\tau=t} \right\| = \alpha_2$$

for all $t \geq t_0$. In addition to that, $\mathbf{f}(t) = \mathbf{0}$ and $\left\| \frac{d^2}{d\tau^2}\mathbf{f}(\tau) \right\|$ has an upper bound, which does not depend on t. Therefore, using Proposition 4.2 twice, it follows that

$$\underset{\substack{\delta_2 > 0 \\ \beta_2 > 0}}{\exists} \quad \underset{t \geq t_0}{\forall} \quad : \quad \mathbf{d}^T \mathcal{W}_{\bar{2}}(t, t+\delta_2)\,\mathbf{d} \geq \beta_2.$$

Finally, consider the last case where $\mathbf{d}_1 = \mathbf{d}_2 = \mathbf{0}$ and therefore $\|\mathbf{d}_3\| = 1$. Then,

$$\left\| \left. \frac{d^2}{d\tau^2}\mathbf{f}(\tau) \right|_{\tau=t} \right\| = 1$$

for all $t \geq t_0$ and again, as $\left\| \frac{d^3}{d\tau^3}\mathbf{f}(\tau) \right\|$ is bounded from above, uniformly in t, and

$$\mathbf{f}(t) = \left. \frac{d}{d\tau}\mathbf{f}(\tau) \right|_{\tau=t} = \left. \frac{d^2}{d\tau^2}\mathbf{f}(\tau) \right|_{\tau=t} = \mathbf{0}$$

for all $t \geq t_0$, it follows, using Proposition 4.2 twice, that

$$\underset{\substack{\delta_3 > 0 \\ \beta_3 > 0}}{\exists} \quad \underset{t \geq t_0}{\forall} \quad : \quad \mathbf{d}^T \mathcal{W}_{\bar{2}}(t, t+\delta_3)\,\mathbf{d} \geq \beta_3.$$

Either way,

$$\underset{\substack{\delta > 0 \\ \beta > 0}}{\exists} \quad \underset{t \geq t_0}{\forall} \quad \underset{\substack{\mathbf{d} \in \mathbb{R}^9 \\ \|\mathbf{d}\| = 1}}{\forall} \quad : \quad \mathbf{d}^T \mathcal{W}_{\bar{2}}(t, t+\delta)\,\mathbf{d} \geq \beta,$$

which means that (12) is uniformly completely observable. As the LTV systems (7) and (12) are related by a Lyapunov transformation, it follows that (7) is also uniformly completely observable. □

4.3. Dynamic accelerometer bias estimation

This section presents observability conditions for dynamic accelerometer bias estimation. Before going into the details, some straightforward but very useful and inspiring properties regarding the observability of the system are presented and discussed.

In compact form, the dynamic system (8) can be written as

$$\begin{cases} \dot{\mathbf{x}}_3(t) = \mathbf{A}_3(t)\mathbf{x}_3(t) + \mathbf{B}_3\mathbf{u}_3(t) \\ \mathbf{y}_3(t) = \mathbf{C}_3\mathbf{x}_3(t) \end{cases}, \tag{13}$$

where $\mathbf{u}_3(t) = \mathbf{a}(t)$ is the input of the system,

$$\mathbf{x}_3(t) = \begin{bmatrix} \mathbf{v}(t) \\ \mathbf{g}(t) \\ \mathbf{b}(t) \end{bmatrix} \in \mathbb{R}^9$$

is the vector of states of the system,

$$\mathbf{A}_3(t) = \begin{bmatrix} -\mathbf{S}[\omega(t)] & \mathbf{I} & -\mathbf{I} \\ \mathbf{0} & -\mathbf{S}[\omega(t)] & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix},$$

$$\mathbf{B}_3 = \begin{bmatrix} \mathbf{I} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

and

$$\mathbf{C}_3 = [\mathbf{I}\ \mathbf{0}\ \mathbf{0}].$$

Within this framework, suppose that the angular velocity $\boldsymbol{\omega}(t)$ is constant. In this situation, the dynamic system (13) is LTI and thus, to assess the observability of the system, it suffices to check the rank of the observability matrix $\boldsymbol{O}_3$ associated to the pair $(\mathbf{A}_3, \mathbf{C}_3)$,

$$\boldsymbol{O}_3 := \begin{bmatrix} \mathbf{C}_3 \\ \mathbf{C}_3\mathbf{A}_3 \\ \mathbf{C}_3\mathbf{A}_3^2 \\ \cdots \\ \mathbf{C}_3\mathbf{A}_3^{n-1} \end{bmatrix}. \tag{14}$$

After a few algebraic manipulations it is possible to write (14) as

$$\boldsymbol{O}_3 = \mathbf{D}_{\mathbf{v}8} \begin{bmatrix} \mathbf{I} & \mathbf{0} & \mathbf{0} \\ -\boldsymbol{\Lambda} & \mathbf{I} & -\mathbf{I} \\ \boldsymbol{\Lambda}^2 & -2\boldsymbol{\Lambda} & \boldsymbol{\Lambda} \\ -\boldsymbol{\Lambda}^3 & 3\boldsymbol{\Lambda}^2 & -\boldsymbol{\Lambda}^2 \\ \boldsymbol{\Lambda}^4 & -4\boldsymbol{\Lambda}^3 & \boldsymbol{\Lambda}^3 \\ \vdots & \vdots & \vdots \\ \boldsymbol{\Lambda}^8 & -8\boldsymbol{\Lambda}^7 & \boldsymbol{\Lambda}^7 \end{bmatrix} \mathbf{D}_{\mathbf{v}3}^*,$$

where $\mathbf{D}_{\mathbf{v}8} := \operatorname{diag}(\mathbf{V}, \ldots, \mathbf{V})$, $\mathbf{D}_{\mathbf{v}3}^* := \operatorname{diag}(\mathbf{V}^*, \mathbf{V}^*, \mathbf{V}^*)$, $\mathbf{V}$ is a unitary matrix, i.e., $\mathbf{V} \in \left\{\mathbf{X} \in \mathbb{R}^{3\times3} : \mathbf{X}^T\mathbf{X} = \mathbf{I}\right\}$, and

$$\boldsymbol{\Lambda} = \begin{bmatrix} \|\omega\|\,j & 0 & 0 \\ 0 & -\|\omega\|\,j & 0 \\ 0 & 0 & 0 \end{bmatrix}.$$

Thus, it is immediate to conclude that

- for $\boldsymbol{\omega} = \mathbf{0}$, $\operatorname{rank}[\boldsymbol{O}_3] = 6$ and
- for $\boldsymbol{\omega} \neq \mathbf{0}$, $\operatorname{rank}[\boldsymbol{O}_3] = 8$.

From this first result it is already possible to say that the system (13) is not observable for, at least, some trajectories of $\boldsymbol{\omega}(t)$, and this is not a surprise. Indeed, for $\boldsymbol{\omega}(t) = \mathbf{0}$, both the gravity and the bias are constant in body-fixed coordinates (and inertial coordinates too) and it is impossible to distinguish between them solely based on the velocity measurements. However, in this situation, it is straightforward to show that it would be possible to design an observer for both $\mathbf{v}(t)$ and the quantity $\mathbf{g}(t) - \mathbf{b}(t)$. When $\boldsymbol{\omega}$ is constant but nonzero, the degree of observability of the system increases. In this situation it is also straightforward to show that the non-observable subspace is given by

$$\begin{bmatrix} \mathbf{v} \\ \mathbf{g} \\ \mathbf{b} \end{bmatrix} = \operatorname{span}\left(\begin{bmatrix} \mathbf{0} \\ \boldsymbol{\omega} \\ \boldsymbol{\omega} \end{bmatrix}\right).$$

Thus, it is still possible to estimate both $\mathbf{v}(t)$ and $\mathbf{g}(t) - \mathbf{b}(t)$. This fact is important and will be exploited shortly as it suggests that $\mathbf{g}(t) - \mathbf{b}(t)$ is observable regardless of the trajectory described by the angular velocity. Also, since the non-observable subspace for constant non-null angular velocity is related to the axis of rotation, it is expectable that, if the axis of rotation changes, the system becomes observable. Before presenting the main results, which confirm this conjecture, a Lyapunov state transformation is introduced that greatly simplifies the analysis of the system.

In Section 4.1 the observability of the system was assessed through the use of an orthogonal Lyapunov transformation that renders the pair $\left(\mathbf{A}_{\bar{1}}, \mathbf{C}_{\bar{1}}\right)$ time invariant. Although the application of this technique to (13) does not render the pair $\left(\mathbf{A}_{\bar{3}}(t), \mathbf{C}_{\bar{3}}(t)\right)$ time invariant, it is still useful as it reduces the number of time-varying elements of the new dynamics. Coupled with this, it has been shown that both $\mathbf{v}(t)$ and $\mathbf{g}(t) - \mathbf{b}(t)$ are observable for constant angular velocities. These two ideas motivate the state transformation

$$\mathbf{x}_{\bar{3}}(t) := \mathbf{T}_3(t)\mathbf{x}_3(t), \tag{15}$$

with

$$\mathbf{T}_3(t) := \begin{bmatrix} \mathbf{R}(t) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}(t) & -\mathbf{R}(t) \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}.$$

Notice that (15) is a Lyapunov state transformation as

- $\mathbf{T}_3(t)$ is continuously differentiable for all t;
- Both $\mathbf{T}_3(t)$ and $\dot{\mathbf{T}}_3(t)$ are bounded for all t, where
$$\dot{\mathbf{T}}_3(t) = \begin{bmatrix} \mathbf{R}(t)\mathbf{S}\left[\omega(t)\right] & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}(t)\mathbf{S}\left[\omega(t)\right] & -\mathbf{R}(t)\mathbf{S}\left[\omega(t)\right] \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix};$$
- $\det\left[\mathbf{T}_3(t)\right] = 1$.

The fact that (15) is a Lyapunov transformation suffices to establish the equivalence of observability properties between $\mathbf{x}_3(t)$ and $\mathbf{x}_{\bar{3}}(t)$.

The dynamics of $\mathbf{x}_{\bar{3}}$ are given by

$$\begin{cases} \dot{\mathbf{x}}_{\bar{3}}(t) = \mathbf{A}_{\bar{3}}(t)\mathbf{x}_{\bar{3}}(t) + \mathbf{B}_{\bar{3}}(t)\mathbf{u}_3(t) \\ \mathbf{y}_3(t) = \mathbf{C}_{\bar{3}}(t)\mathbf{x}_{\bar{3}}(t) \end{cases}, \tag{16}$$

where

$$\mathbf{A}_{\bar{3}}(t) = \begin{bmatrix} \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & -\mathbf{R}(t)\mathbf{S}\left[\omega(t)\right] \\ \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix}, \ \mathbf{B}_{\bar{3}}(t) = \begin{bmatrix} \mathbf{R}(t) \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

and $\mathbf{C}_{\bar{3}}(t) = \left[\mathbf{R}^T(t)\ \mathbf{0}\ \mathbf{0}\right]$. It is a simple matter of computation to show that the transition matrix associated with $\mathbf{A}_{\bar{3}}(t)$ is given by

$$\boldsymbol{\phi}_{\bar{3}}(t, t_0) = \begin{bmatrix} \mathbf{I} & (t - t_0)\,\mathbf{I} & (t - t_0)\,\mathbf{R}(t_0) - \mathbf{R}^{[1]}(t, t_0) \\ \mathbf{0} & \mathbf{I} & \mathbf{R}(t_0) - \mathbf{R}(t) \\ \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}$$

and, if $\mathcal{W}_{\bar{3}}\left(t_0, t_f\right)$ denotes the observability Gramian associated with the pair $\left(\mathbf{A}_{\bar{3}}(t), \mathbf{C}_{\bar{3}}(t)\right)$,

$$\mathbf{d}^T \mathcal{W}_{\bar{3}}\left(t_0, t_f\right)\mathbf{d} = \\ = \int_{t_0}^{t_f} \left\|\mathbf{d}_1 + (\tau - t_0)\,\mathbf{d}_2 + (\tau - t_0)\,\mathbf{R}(t_0)\,\mathbf{d}_3 - \mathbf{R}^{[1]}(\tau, t_0)\,\mathbf{d}_3\right\|^2 d\tau$$

for all

$$\mathbf{d} = \begin{bmatrix} \mathbf{d}_1 \\ \mathbf{d}_2 \\ \mathbf{d}_3 \end{bmatrix} \in \mathbb{R}^9, \ \|\mathbf{d}\| = 1.$$

The following theorem provides a necessary and sufficient condition for the observability of (8).

Theorem 4.4. *The LTV system* (8) *is observable on* $\left[t_0, t_f\right]$ *if and only if the direction of the angular velocity* $\boldsymbol{\omega}(t)$ *changes for some* $t_1 \in \left[t_0, t_f\right]$ *or, equivalently,*

$$\underset{\substack{\mathbf{d} \in \mathbb{R}^3 \\ \|\mathbf{d}\| = 1}}{\forall} \quad \underset{t_1 \in \left[t_0, t_f\right]}{\exists} \quad : \quad \mathbf{S}\left[\omega(t_1)\right]\mathbf{d} \neq \mathbf{0}. \tag{17}$$

Proof. Let

$$\mathbf{f}(\tau) := \mathbf{d}_1 + (\tau - t_0)\,\mathbf{d}_2 + (\tau - t_0)\,\mathbf{R}(t_0)\,\mathbf{d}_3 - \mathbf{R}^{[1]}(\tau, t_0)\,\mathbf{d}_3$$

and notice that

$$\mathbf{d}^T \mathcal{W}_{\bar{3}}\left(t_0, t_f\right)\mathbf{d} = \int_{t_0}^{t_f} \|\mathbf{f}(\tau)\|^2\, d\tau.$$

If $\mathbf{d}_1 \neq \mathbf{0}$ then

$$\|\mathbf{f}(t_0)\|^2 = \|\mathbf{d}_1\|^2 = \alpha_1^2 > 0.$$

Moreover, notice that $\frac{d}{d\tau}\|\mathbf{f}(\tau)\|^2$ is a continuous function and therefore it has an upper bound on any non-empty limited closed interval (Weierstrass Theorem). As, in addition to that,

$$\mathbf{d}^T \mathcal{W}_{\bar{3}}(t_0, t_0)\,\mathbf{d} = \int_{t_0}^{t_0} \|\mathbf{f}(\tau)\|^2\, d\tau = 0,$$

it follows, using Proposition 4.2, that

$$\underset{\substack{0 < \delta_1 \leq t_f - t_0 \\ \beta_1 > 0}}{\exists} \quad : \quad \mathbf{d}^T \mathcal{W}_{\bar{3}}(t_0, t_0 + \delta_1)\,\mathbf{d} \geq \beta_1.$$

Suppose now that $\mathbf{d}_1 = \mathbf{0}$ and $\mathbf{d}_2 \neq \mathbf{0}$. Then,

$$\left\| \left.\frac{d}{d\tau}\mathbf{f}(\tau)\right|_{\tau = t_0} \right\|^2 = \|\mathbf{d}_2\|^2 = \alpha_2^2 > 0$$

and, using Proposition 4.2 again, it is immediate to show that

$$\underset{\substack{0 < \delta_2 \leq t_f - t_0 \\ \beta_2 > 0}}{\exists} \quad : \quad \mathbf{d}^T \mathcal{W}_{\bar{3}}(t_0, t_0 + \delta_2)\,\mathbf{d} \geq \beta_2.$$

Consider now the last case where $\mathbf{d}_1 = \mathbf{d}_2 = \mathbf{0}$ and therefore $\|\mathbf{d}_3\| = 1$. It is easy to see that

$$\left\| \frac{d^2}{d\tau^2}\mathbf{f}(\tau) \right\| = \|\mathbf{S}\left[\omega(\tau)\right]\mathbf{d}_3\|$$

Now, using (17), it is possible to write

$$\underset{\substack{t_1 \in [t_0, t_f] \\ \alpha_3 > 0}}{\exists} : \left\| \frac{d^2}{d\tau}\mathbf{f}(\tau)\Big|_{\tau=t_1} \right\| = \alpha_3$$

for all $\mathbf{d}_3$ such that $\|\mathbf{d}_3\| = 1$. But then, using Proposition 4.2 again, it follows, again, that

$$\underset{\substack{0<\delta_3 \leq t_f - t_0 \\ \beta_3 > 0}}{\exists} : \mathbf{d}^T \mathcal{W}_{\bar{3}}(t_0, t_0+\delta_3)\,\mathbf{d} \geq \beta_3.$$

Therefore, if (17) is true, the LTVS is observable on $[t_0, t_f]$ and, as (8) and (13) are related through a Lyapunov transformation, it follows that (8) is also observable on $[t_0, t_f]$. Suppose now that (17) is not true. Then,

$$\underset{\substack{\mathbf{d}_0 \in \mathbb{R}^3 \\ \|\mathbf{d}_0\|=1}}{\exists} \quad \underset{t \in [t_0, t_f]}{\forall} : \mathbf{S}(\omega(t))\,\mathbf{d}_0 = \mathbf{0}.$$

Let

$$\mathbf{d} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{d}_0 \end{bmatrix}.$$

Then, it is straightforward to show that

$$\underset{t \in [t_0, t_f]}{\forall} : \mathbf{f}(t) = \mathbf{0}$$

and therefore

$$\underset{\substack{\mathbf{d} \in \mathbb{R}^n \\ \|\mathbf{d}\| \in \mathbb{R}^9}}{\exists} : \mathbf{d}^T \mathcal{W}_{\bar{3}}(t_0, t_f)\,\mathbf{d} = 0,$$

which means that (13) is not observable on $[t_0, t_f]$. Thus, if (13) is observable on $[t_0, t_f]$, it follows that (17) is true. As (8) and (13) are related through a Lyapunov transformation, one is observable if and only if so is the other. Therefore, if (8) is observable on $[t_0, t_f]$, it follows that (17) is true, which concludes the proof. □

The following theorem, that provides a necessary and sufficient condition for a stronger form of observability, is the main result of this section.

Theorem 4.5. *The LTV system* (8) *is uniformly completely observable if and only if*

$$\underset{\substack{\delta>0 \\ \epsilon>0}}{\exists} \quad \underset{t \geq t_0}{\forall} \quad \underset{\substack{\mathbf{d} \in \mathbb{R}^3 \\ \|\mathbf{d}\|=1}}{\forall} \quad \underset{t_1 \in [t, t+\delta]}{\exists} : \|\mathbf{S}(\omega(t_1))\,\mathbf{d}\| \geq \epsilon. \tag{18}$$

Proof. The proof of sufficiency follows steps similar to those presented in the proof of Theorem 4.3 and therefore it is omitted. Suppose now that (18) is not true. Then,

$$\underset{\substack{\delta>0 \\ \epsilon>0}}{\forall} \quad \underset{t_1 \geq t_0}{\exists} \quad \underset{\substack{\mathbf{d}_0 \in \mathbb{R}^3 \\ \|\mathbf{d}_0\|=1}}{\exists} \quad \underset{t \in [t_1, t_1+\delta]}{\forall} : \|\mathbf{S}(\omega(t))\,\mathbf{d}\| < \epsilon. \tag{19}$$

Let

$$\mathbf{d} = \begin{bmatrix} \mathbf{0} \\ \mathbf{0} \\ \mathbf{d}_0 \end{bmatrix}.$$

Then,

$$\mathbf{d}^T \mathcal{W}_{\bar{3}}(t_1, t_1+\delta)\,\mathbf{d} = \int_{t_1}^{t_1+\delta} \left\|(\tau - t_1)\,\mathbf{R}(t_1)\,\mathbf{d}_0 - \mathbf{R}^{[1]}(\tau, t_1)\,\mathbf{d}_0\right\|^2 d\tau,$$

which may be rewritten as

$$\begin{aligned}
&\mathbf{d}^T \mathcal{W}_{\bar{3}}(t_1, t_1+\delta)\,\mathbf{d} = \\
&= \int_{t_1}^{t_1+\delta} \left\| \int_{t_1}^{\tau} [\mathbf{R}(t_1) - \mathbf{R}(\sigma_1)]\,\mathbf{d}_0 d\sigma_1 \right\|^2 d\tau \\
&= \int_{t_1}^{t_1+\delta} \left\| \int_{t_1}^{\tau} \left(\mathbf{R}(t_1) - \left[\mathbf{R}(t_1) + \int_{t_0}^{\sigma_1} \dot{\mathbf{R}}(\sigma_2)\, d\sigma_2\right]\right)\mathbf{d}_0 d\sigma_1 \right\|^2 dt \\
&= \int_{t_1}^{t_1+\delta} \left\| \int_{t_1}^{\tau} \int_{t_1}^{\sigma_1} \dot{\mathbf{R}}(d\sigma_2)\,\mathbf{d}_0 d\sigma_2 d\sigma_1 \right\|^2 d\tau.
\end{aligned}$$

Substituting the dynamics of the rotation matrix gives

$$\begin{aligned}
&\mathbf{d}^T \mathcal{W}_{\bar{3}}(t_1, t_1+\delta)\,\mathbf{d} = \\
&= \int_{t_1}^{t_1+\delta} \left\| \int_{t_1}^{t} \int_{t_1}^{\sigma_1} \mathbf{R}(\sigma_2)\,\mathbf{S}[\omega(\sigma_2)]\,\mathbf{d}_0 d\sigma_2 d\sigma_1 \right\|^2 d\tau.
\end{aligned} \tag{20}$$

Using simple norm inequalities in (20) gives

$$\begin{aligned}
&\mathbf{d}^T \mathcal{W}_{\bar{3}}(t_1, t_1+\delta)\,\mathbf{d} \leq \\
&\leq \int_{t_1}^{t_1+\delta} \int_{t_1}^{t} \int_{t_1}^{\sigma_1} \|\mathbf{R}(\sigma_2)\,\mathbf{S}[\omega(\sigma_2)]\,\mathbf{d}_0\|^2\, d\sigma_2 d\sigma_1 d\tau
\end{aligned}$$

and, as the rotation has unit norm,

$$\begin{aligned}
&\mathbf{d}^T \mathcal{W}_{\bar{3}}(t_1, t_1+\delta)\,\mathbf{d} \leq \\
&\leq \int_{t_1}^{t_1+\delta} \int_{t_1}^{t} \int_{t_1}^{\sigma_1} \|\mathbf{S}[\omega(\sigma_2)]\,\mathbf{d}_0\|^2\, d\sigma_2 d\sigma_1 d\tau.
\end{aligned} \tag{21}$$

Using (19) in (21) allows to conclude that, for all $\delta > 0$ and $\epsilon > 0$,

$$\underset{t_1 \geq t_0}{\exists} \quad \underset{\substack{\mathbf{d} \in \mathbb{R}^9 \\ \|\mathbf{d}\|=1}}{\exists} \quad \underset{t \in [t_1, t_1+\delta]}{\forall} : \mathbf{d}^T \mathcal{W}_{\bar{3}}(t_1, t_1+\delta)\,\mathbf{d} \leq \frac{\delta^3}{6}\epsilon^2,$$

which implies that the LTV system (13) is not uniformly completely observable. Therefore, if (13) is uniformly completely observable, then (18) is true. Finally, as (8) and (13) are related through a Lyapunov state transformation, it follows that if (8) is uniformly completely observable, then (18) is true, which completes the proof. □

Remark 3. The meaning of the technical condition stated in Theorem 4.5 is not evident at first glance. To make it clear notice that, when (18) is not satisfied, the direction of the angular velocity is converging to a constant vector. While for observability it suffices that the direction of the angular velocity changes, for uniform complete observability a minimum level of excitation is required. This is reflected as the requirement of a minimum change of the direction of the angular velocity vector.

4.4. Navigation with dynamic accelerometer bias determination in the presence of unknown gravity

This section presents the last result of the paper, which assesses the observability of a navigation system with dynamic accelerometer bias estimation. This result is closely related to the one presented in Section 4.3, since the nominal dynamics for navigation with dynamic accelerometer bias determination can be regarded as an extension of the dynamics for dynamic accelerometer bias estimation.

The first result presented in this section provides a necessary and sufficient condition for the observability of (9).

Theorem 4.6. *The LTV system* (9) *is observable on* $\left[t_0, t_f\right]$ *if and only if* (17) *holds.*

Proof. The system dynamics (9) can be rewritten, in compact form, as

$$\begin{cases} \dot{\mathbf{x}}_4(t) = \mathbf{A}_4(t)\mathbf{x}_4(t) + \mathbf{B}_4\mathbf{u}_4(t) \\ \mathbf{y}_4(t) = \mathbf{C}_4\mathbf{x}_4(t) \end{cases},$$

where $\mathbf{u}_4(t) = \mathbf{a}(t)$ is the input of the system,

$$\mathbf{x}_4(t) = \begin{bmatrix} \mathbf{p}(t) \\ \mathbf{v}(t) \\ \mathbf{g}(t) \\ \mathbf{b}(t) \end{bmatrix} \in \mathbb{R}^{12}$$

is the vector of states of the system,

$$\mathbf{A}_4(t) = \begin{bmatrix} -\mathbf{S}\left[\omega(t)\right] & -\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & -\mathbf{S}\left[\omega(t)\right] & \mathbf{I} & -\mathbf{I} \\ \mathbf{0} & \mathbf{0} & -\mathbf{S}\left[\omega(t)\right] & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix},$$

$$\mathbf{B}_4 = \begin{bmatrix} \mathbf{0} \\ \mathbf{I} \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

and $\mathbf{C}_4 = [\mathbf{I}\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}]$. Consider the Lyapunov transformation

$$\mathbf{x}_{\bar{4}}(t) := \mathbf{T}_4(t)\mathbf{x}_4(t), \tag{22}$$

with

$$\mathbf{T}_4(t) := \begin{bmatrix} \mathbf{R}(t) & \mathbf{0} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{R}(t) & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{R}(t) & -\mathbf{R}(t) \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{I} \end{bmatrix}.$$

Then, the new system dynamics can be written as

$$\begin{cases} \dot{\mathbf{x}}_{\bar{4}}(t) = \mathbf{A}_{\bar{4}}(t)\mathbf{x}_{\bar{4}}(t) + \mathbf{B}_{\bar{4}}(t)\mathbf{u}_4(t) \\ \mathbf{y}_{\bar{4}}(t) = \mathbf{C}_{\bar{4}}(t)\mathbf{x}_{\bar{4}}(t) \end{cases}, \tag{23}$$

where

$$\mathbf{A}_{\bar{4}}(t) = \begin{bmatrix} \mathbf{0} & -\mathbf{I} & \mathbf{0} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{I} & \mathbf{0} \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & -\mathbf{R}(t)\mathbf{S}\left[\omega(t)\right] \\ \mathbf{0} & \mathbf{0} & \mathbf{0} & \mathbf{0} \end{bmatrix},$$

$$\mathbf{B}_{\bar{4}}(t) = \begin{bmatrix} \mathbf{0} \\ \mathbf{R}(t) \\ \mathbf{0} \\ \mathbf{0} \end{bmatrix},$$

and $\mathbf{C}_{\bar{4}}(t) = \left[\mathbf{R}^T(t)\ \mathbf{0}\ \mathbf{0}\ \mathbf{0}\right]$. The fact that (22) is a Lyapunov transformation suffices to establish the equivalence of observability properties between $\mathbf{x}_4$ and $\mathbf{x}_{\bar{4}}$. The similarities between (23) and (16) are obvious. There is, in fact, just an extra level of integrators. The remainder of the proof follows the same steps as in Theorem 4.4 and is therefore omitted. □

The following theorem is the main result of this section and provides a necessary and sufficient condition for the system (9) to be uniformly completely observable.

Theorem 4.7. *The dynamic system* (9) *is uniformly completely observable if and only if* (18) *holds.*

Proof. The proof follows the same steps as in Theorem 4.5 and is therefore omitted. □

5. Conclusions

Navigation Systems are key elements of a large variety of robotic systems. This paper provided observability results regarding systems related to the estimation of linear motion quantities of mobile platforms (position, linear velocity, linear acceleration, and accelerometer bias), in 3-D, assuming exact angular measurements. Four different cases were studied: i) a simple calibrated sensor suite consisting of an IMU and a positioning sensor. It was shown that the system is not only observable but also uniformly completely observable, even without the knowledge of the acceleration of gravity; ii) a triad of accelerometers with unknown biases but considering that the acceleration of gravity is known. It was shown that this system is also observable and uniformly completely observable; iii) dynamic accelerometer bias estimation. In this case it was proved that not all trajectories yield the system observable. In particular, it was shown that the trajectories should be rich enough in what concerns the evolution of the direction of the angular velocity and, for uniform complete observability to be attained, the direction of the angular velocity cannot stay indefinitely arbitrarily close to a constant vector; and iv) the last case addressed the most general setup where the triad of accelerometers may have an unknown bias and the gravity is also supposed to be unknown. It was shown that the system is observable if and only if the attitude evolution is sufficiently rich, in the same sense as the one presented for dynamic accelerometer bias estimation. Moreover, it was also shown that the system is uniformly completely observable if and only if a persistent change in the direction of the angular velocity occurs. The summary of the conclusions is presented in Table 1.

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Scenario	System dynamics	Available quantities	Estimated quantities	Observable	Uniformly completely observable
Navigation with calibrated accelerometer	(6)	$(\mathbf{p}, \mathbf{a}, \mathbf{b})$	$(\mathbf{p}, \mathbf{v}, \mathbf{g})$	yes	yes
Navigation with known gravity	(7)	$(\mathbf{p}, \mathbf{a}, \mathbf{g})$	$(\mathbf{p}, \mathbf{v}, \mathbf{b})$	yes	yes
Dynamic accelerometer bias estimation	(8)	$(\mathbf{v}, \mathbf{a})$	$(\mathbf{v}, \mathbf{g}, \mathbf{b})$	if and only if (17) is true	if and only if (18) is true
Navigation with gravity and accelerometer bias dynamic estimation	(9)	$(\mathbf{p}, \mathbf{a})$	$(\mathbf{p}, \mathbf{v}, \mathbf{g}, \mathbf{b})$	if and only if (17) is true	if and only if (18) is true

Table 1: Summary of observability conclusions.

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A. Proof of Proposition 4.2

Proposition. *Let* $\mathbf{f}(t) : \left[t_0, t_f\right] \subset \mathbb{R} \rightarrow \mathbb{R}^n$ *be a continuous and i-times continuously differentiable function on* $\mathcal{I} := \left[t_0, t_f\right]$, $T := t_f - t_0 > 0$, *and such that*

$$\mathbf{f}(t_0) = \dot{\mathbf{f}}(t_0) = \ldots = \mathbf{f}^{(i-1)}(t_0) = \mathbf{0}.$$

Further assume that

$$\max_{t \in \mathcal{I}} \left\|\mathbf{f}^{(i+1)}(t)\right\| \leq C. \tag{24}$$

If

$$\exists_{\substack{\alpha > 0 \\ t_1 \in \mathcal{I}}} : \left\|\mathbf{f}^{(i)}(t_1)\right\| \geq \alpha, \tag{25}$$

then

$$\exists_{\substack{0 < \delta \leq T \\ \beta > 0}} : \left\|\mathbf{f}(t_0 + \delta)\right\| \geq \beta. \tag{26}$$

Proof. Firstly, notice that the case $C = 0$ is trivial. Indeed, if $C = 0$, then

$$\forall_{t \in \mathcal{I}} : \mathbf{f}^{(i)}(t) = \mathbf{f}^{(i)}(t_1)$$

and therefore

$$\mathbf{f}(t_0 + \delta) = \mathbf{f}^{(i)}(t_1) \int_{t_0}^{t_0+\delta} \int_{t_0}^{\sigma_1} \ldots \int_{t_0}^{\sigma_{i-1}} d\sigma_i \ldots d\sigma_1 = \frac{\delta^i}{i!} \mathbf{f}^{(i)}(t_1),$$

which implies (26). The remainder of the proof considers $C > 0$. Suppose that (24) and (25) are true. Then, using simple norm inequalities, it is possible to write

$$\left\|\mathbf{f}^{(i)}(t_1)\right\|_\infty \geq \frac{1}{\sqrt{n}} \alpha$$

and

$$\max_{t\in\mathcal{I}} \left\|\dot{\mathbf{f}}^{(i+1)}(t)\right\|_\infty \leq C.$$

Let

$$k := \arg\max_{j=1,\ldots,n} \left|f_j^{(i)}(t_1)\right|,$$

where

$$\mathbf{f}^{(i)}(t) = \begin{bmatrix} f_1^{(i)}(t) \\ \vdots \\ f_n^{(i)}(t) \end{bmatrix}.$$

Evidently,

$$\left|f_k^{(i)}((t_1)\right| \geq \frac{1}{\sqrt{n}}\alpha$$

and

$$\max_{t\in\mathcal{I}} \left|\dot{f}_k^{(i+1)}(t)\right| \leq C. \tag{27}$$

Resorting to the Lagrange's Theorem, it is possible to write that

$$\left|f_k^{(i)}(t) - f_k^{(i)}(t_1)\right| = \left|f_k^{(i+1)}(\xi(t))(t - t_1)\right| \tag{28}$$

for all $t \in \mathcal{I}$, where $\xi(t) \in \left]t_0, t_f\right[$. Using simple norm inequalities and (27) in (28) gives

$$\left|f_k^{(i)}(t) - f_k^{(i)}(t_1)\right| \leq C\,|t - t_1|$$

and therefore

$$f_k^{(i)}(t) \geq f_k^{(i)}(t_1) - C\,|t - t_1|$$

for all $t \in \mathcal{I}$. Now assume, without loss of generality, that $f_k^{(i)}(t_1) > 0$. Then, there exists an interval $\mathcal{I}_1 = [t_2, t_3] \subset \mathcal{I}$, $t_2 < t_3$, where either $t_2 = t_1$ or $t_3 = t_1$, and with length

$$T_1 := \frac{1}{2}\min\left(T, \frac{\alpha}{\sqrt{n}C}\right),$$

such that

$$\underset{t\in\mathcal{I}_1}{\forall} \;:\; f_k^{(i)}(t) \geq f_k^{(i)}(t_1) - C\,|t - t_1| > 0. \tag{29}$$

Integrating (29) on $\mathcal{I}_1$ gives

$$\int_{\mathcal{I}_1} f_k^{(i)}(t)\,dt \geq \beta > 0,$$

where

$$\beta := T_1\left(\frac{\alpha}{\sqrt{n}} - \frac{CT_1}{2}\right) > 0.$$

Now, notice that

$$f_k^{(i-1)}(t_3) = \int_{t_0}^{t_3} f_k^{(i)}(t)\,dt = \int_{t_0}^{t_2} f_k^{(i)}(t)\,dt + \int_{\mathcal{I}_1} f_k^{(i)}(t)\,dt.$$

If

$$f_k^{(i-1)}(t_3) \neq 0$$

then

$$\underset{\substack{0<\delta_1\leq T\\ \beta_1>0}}{\exists} \;:\; \left|f_k^{(i-1)}(t_0 + \delta_1)\right| \geq \beta_1.$$

Otherwise, it must be

$$f_k^{(i-1)}(t_2) = -\int_{\mathcal{I}_1} f_k^{(i)}(t)\,dt \neq 0,$$

which implies that

$$\underset{\substack{0<\delta_2<T\\ \beta_2>0}}{\exists} \;:\; \left|f_k^{(i-1)}(t_0 + \delta_2)\right| \geq \beta_2.$$

Either way,

$$\underset{\substack{0<\delta_3\leq T\\ \beta_3>0}}{\exists} \;:\; \left|f_k^{(i-1)}(t_0 + \delta_3)\right| \geq \beta_3.$$

Repeating the same argument $i-1$ times, it is immediate to show that

$$\underset{\substack{0<\delta<T\\ \beta>0}}{\exists} \;:\; |f_k(t_0 + \delta)| \geq \beta$$

and, using simple norm inequalities

$$\underset{\substack{0<\delta<T\\ \beta>0}}{\exists} \;:\; \|\mathbf{f}(t_0 + \delta)\| \geq \beta,$$

which concludes the proof. □