# Survey of Maneuvering Target Tracking. Part I: Dynamic Models

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**This is the first part of a comprehensive and up-to-date survey of the techniques for tracking maneuvering targets without addressing the so-called measurement-origin uncertainty. It surveys various mathematical models of target motion/dynamics proposed for maneuvering target tracking, including 2D and 3D maneuver models as well as coordinate-uncoupled generic models for target motion. This survey emphasizes the underlying ideas and assumptions of the models. Interrelationships among models and insight to the pros and cons of models are provided. Some material presented here has not appeared elsewhere.**

**CONTENTS**



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## I. INTRODUCTION

The key to successful target tracking lies in the effective extraction of useful information about the target's state from observations. A good model of the target will certainly facilitate this information extraction to a great extent. In general, one can say without exaggeration that a good model is worth a thousand pieces of data. This statement has an even stronger positive connotation in target tracking where observation data are rather limited. Most tracking algorithms are model based because knowledge of target motion is available and a good model-based tracking algorithm will greatly outperform any model-free tracking algorithm if the underlying model turns out to be a good one. As such, it is hard to overstate the importance of the role of a good model here.

Various mathematical models of target motion have been developed over the past three decades. They are, however, scattered in the literature. Many of them have never appeared in any periodical in the public domain. As a result, few people have a good knowledge of these models. This is partly due to a lack of a comprehensive survey. The importance of such a survey for both practitioners and researchers in the tracking community is evident. The single best source so far is, in our opinion, the recent book by Blackman and Popoli [1], which is nonetheless far from complete. Some more or less standard models for target motion can be found in established books on target tracking and/or estimation, such as [2–12].

This paper is the first part of a comprehensive and up-to-date survey of the techniques for maneuvering target tracking. The survey is an ongoing project. The conference versions of its first several parts have appeared in [13–17]. It is well known that the so-called measurement-origin uncertainty and target motion uncertainty are two major challenges in target tracking. To limit the scope of the work, this survey deals only with the second uncertainty, leaving the techniques unique for the data-association problems untouched.

Target detection, tracking, and recognition are closely interrelated areas, with significant overlaps. It is not easy to draw a clear line to separate them. To be relatively more focused, this part covers mainly dynamic models of a "point target," that is, those of the dynamic (temporal) behaviors, rather than spatial characteristics, of a target. While many of these models are also useful for target detection and recognition, this survey is only concerned with their value for target tracking. This of course does not prevent us from developing or applying a model that describes both the temporal evolution and spatial characteristics of a target.

Needless to say, target dynamic models and tracking algorithms have intimate ties. The applicability of a target dynamic model for a practical problem can hardly be evaluated without referring to the corresponding tracking algorithms used. In other words, some target models and tracking algorithms work well jointly. To be more focused and concise, however, this interdependence is largely ignored here.

This survey emphasizes the underlying ideas and assumptions of the models. This should help the reader understand not only how these models work but also their pros and cons. It is hoped that a distinctive feature of this survey is that it reveals well the interrelationships among various models. However, the reader should keep in mind that much of such discussion is based on our personal views and preferences, not always accurate or unbiased, although a great deal of effort has been made toward this goal. In addition to such discussions, some material included in this survey has not appeared elsewhere.

Regrettably, many important issues associated with the target dynamic models, particularly those of implementation, cannot be discussed (at least to a desirable degree) due to space limitation as well as our background and experience. Nevertheless, we would appreciate very much receiving comments and any missing material that should be covered in this survey.

This paper is a heavily revised and extended version of [13]. Its remaining part is organized as follows. Section II gives briefly state-space representations of target dynamics and observation system. Section III describes nonmaneuver models. The presentation of dynamic models of target maneuvers breaks down as follows: while Section IV deals with those that are uncoupled along different spatial coordinates, 2D and 3D coupled models are reviewed in Sections V and VI, respectively. The final section provides concluding remarks.

## II. MATHEMATICAL MODELS FOR MANEUVERING TARGET TRACKING

The primary objective of target tracking is to estimate the state trajectories of a target—a moving or movable object. Although a target is almost never really a point in space and the information about its orientation is valuable for tracking, a target is usually treated as a point object without a shape in tracking, especially in target dynamic models. A target dynamic/motion model describes the evolution of the target state with respect to time.

Almost all maneuvering target tracking methods are model based. They assume that the target motion and its observations can be represented by some known mathematical models sufficiently accurately. The most commonly used such models are those known as state-space models, in the following form of additive noise,

$$x_{k+1} = f_k(x_k, u_k) + w_k \tag{1}$$

$$z_k = h_k(x_k) + v_k \tag{2}$$

where $x_k$, $z_k$, and $u_k$ are the target state, observation, and control input vectors, respectively, at the discrete time $t_k$; $w_k$ and $v_k$ are process and measurement noise sequences, respectively; and $f_k$ and $h_k$ are some vector-valued (possibly time-varying) functions. Such a *discrete-time* model is often obtained[1] by discretizing (sampling) the following *continuous-time* model [2][2]

$$\dot{x}(t) = f(x(t), u(t), t) + w(t), \qquad x(t_0) = x_0 \tag{3}$$

$$z(t) = h(x(t), t) + v(t) \tag{4}$$

where $x_k = x(t_k)$ and it is usually assumed[3] that $z_k = z(t_k), v_k = v(t_k), h_k(x_k) = h(x(t_k), t_k)$. The control input is often assumed (approximately) piecewise constant with $u_k = u(t), t_k \leq t < t_{k+1}$ when discretizing a continuous-time system. In target tracking, the control input $u$ is usually not known. Note that

$$w_k \neq w(t_k), \qquad f_k(x_k, u_k, w_k) \neq f(x(t_k), u(t_k), w(t_k), t_k).$$

In fact, it is often more appropriate to use the following *mixed-time* models for most tracking problems

$$\dot{x}(t) = f(x(t), u(t), t) + w(t), \qquad x(t_0) = x_0 \tag{5}$$

$$z_k = h_k(x_k) + v_k \tag{6}$$

because while observations are usually available only at discrete time instants, the target motion is more accurately modeled in continuous time. For example, target motions should not depend on how and when samples are taken, which is often the case, however, for a discrete-time model. For a similar reason, a discrete-time equivalent model is usually

[1] One so obtained is referred to as a discretized model regardless of its equivalence to the continuous-time system. It is termed a discrete-time equivalent if the discretization is exact and the effects of the continuous-time and discrete-time noise processes are equivalent, which is the case for a linear system without the control input $u$ with the standard discretization (i.e., sample and hold). Note, however, that not all discrete-time models can be obtained by discretizing a continuous-time system. A model defined directly in discrete time following the same principle as one in continuous time is referred to as a direct discrete-time counterpart here. We use the term discrete-time version for either or both of discretized and direct discrete-time models.

[2] We sacrifice rigor for readability: (3) is not actually well defined because $\dot{x} = dx/dt$ may not exist at all since the "continuous-time white noise" has infinite variance and is a specter in the cosmos that represents the nonexistent derivative of a Wiener process and the like. We warn, however, that formal manipulation of (3) or (8) may easily lead to incorrect results. That is why in the mathematical world (3) is replaced by $dx(t) = f(x(t), u(t), t)dt + dw(t)$, $x(t_0) = x_0$ and certain rules must be followed.

[3] Although discrete-time measurement $z_k$ and thus the function $h_k$ and noise $v_k$ are in fact from sensors that do integration over time.

more systematic and consistent, and is in many cases probably preferable to the corresponding direct discrete-time counterpart.

The continuous-, discrete-, and mixed-time linear counterparts of the above models are the corresponding pairs of the following equations

$$x_{k+1} = F_k x_k + E_k u_k + G_k w_k \tag{7}$$

$$\dot{x}(t) = A(t)x(t) + E(t)u(t) + B(t)w(t), \qquad x(t_0) = x_0 \tag{8}$$

$$z_k = H_k x_k + v_k \tag{9}$$

$$z(t) = C(t)x(t) + v(t). \tag{10}$$

One of the major challenges for target tracking arises from the target motion uncertainty. This uncertainty refers to the fact that an accurate dynamic model of the target being tracked is not available to the tracker. Specifically, although the general form of the model (1) or (5) is usually adequate, a tracker lacks knowledge about the actual control input $u$ of the target, and possibly the actual form of $f$, its parameters, or statistical properties of the noise $w$ for the particular target being tracked. Target motion modeling is thus one of the first tasks for maneuvering target tracking. It aims at developing a tractable model that accounts well for the effect of target motion.

In this paper, we describe the efforts and results in modeling the target motion for tracking a maneuvering target without knowing its true dynamic behavior. Most of these efforts have been made along two lines: 1) approximate the actually nonrandom control input $u$ as a random process of certain properties, and 2) describe typical target trajectories by some representative motion models with properly designed parameters.

Target motions are normally classified into two classes: maneuver and nonmaneuver. A nonmaneuvering motion is the *straight and level motion* at a constant velocity[4] in an inertial reference system, sometimes also referred to as the *uniform motion*. Loosely speaking, all other motions belong to the maneuvering mode.

## III. NONMANEUVER MODELS

It is well known that a point moving in our 3D physical world can be described by its 3D position and velocity vectors. For instance,[5] $x = [\mathsf{x}, \dot{\mathsf{x}}, \mathsf{y}, \dot{\mathsf{y}}, \mathsf{z}, \dot{\mathsf{z}}]'$ can be used as a state vector of such a point in the Cartesian coordinate system, where $(\mathsf{x}, \mathsf{y}, \mathsf{z})$ are the position coordinates along $\mathsf{x}$, $\mathsf{y}$, and $\mathsf{z}$ axes, respectively, and $[\dot{\mathsf{x}}, \dot{\mathsf{y}}, \dot{\mathsf{z}}]'$ is the velocity vector. When a target is treated as a point object, the nonmaneuvering motion is thus described by the vector-valued equation $\dot{x}(t) = 0$, where $x = [\dot{\mathsf{x}}, \dot{\mathsf{y}}, \mathsf{z}]'$. Note that $\mathsf{z}$ direction is treated differently because a nonmaneuvering motion is assumed in the horizontal $\mathsf{x}$–$\mathsf{y}$ plane. In practice, this ideal equation is usually modified as $\dot{x}(t) = w(t) \approx 0$, where $w(t)$ is white noise with a "small" effect on $x$ that accounts for unpredictable modeling errors due to turbulence, etc. The corresponding state-space model is given by, with state vector $x = [\mathsf{x}, \dot{\mathsf{x}}, \mathsf{y}, \dot{\mathsf{y}}, \mathsf{z}]'$,

$$\dot{x}(t) = \mathrm{diag}[A_{cv}, 0]x(t) + \mathrm{diag}[B_{cv}, 1]w(t) \tag{11}$$

where $w(t) = [w_x(t), w_y(t), w_z(t)]'$ is a continuous-time vector-valued white noise process with power spectral density matrix $\mathrm{diag}[S_x, S_y, S_z]$, $A_{cv} = \mathrm{diag}[A_2, A_2]$, and $B_{cv} = \mathrm{diag}[B_2, B_2]$ with[6]

$$A_2 = \begin{bmatrix} 0 & 1 \\ 0 & 0 \end{bmatrix}, \qquad B_2 = \begin{bmatrix} 0 \\ 1 \end{bmatrix}. \tag{12}$$

The direct discrete-time counterpart of the above model is [2]

$$\begin{aligned} x_{k+1} &= Fx_k + Gw_k = \mathrm{diag}[F_{cv}, 1]x_k + \mathrm{diag}[G_{cv}, T]w_k \\ &= \mathrm{diag}[F_2, F_2, 1]x_k + \mathrm{diag}[G_2, G_2, T]w_k \end{aligned} \tag{13}$$

where

$$\begin{aligned} &F_{cv} = \mathrm{diag}[F_2, F_2], \qquad G_{cv} = \mathrm{diag}[G_2, G_2] \\ &F_2 = \begin{bmatrix} 1 & T \\ 0 & 1 \end{bmatrix}, \qquad G_2 = \begin{bmatrix} T^2/2 \\ T \end{bmatrix} \end{aligned} \tag{14}$$

$w_k = [w_x, w_y, w_z]'_k$ is a discrete-time white noise sequence and $T$ is the sampling interval. Note that $w_x$ and $w_y$ correspond to noisy "accelerations" along $\mathsf{x}$ and $\mathsf{y}$ axes, respectively, while $w_z$ corresponds to noisy "velocity" along $\mathsf{z}$ axis. If $w$ is uncoupled across its components, then the nonmaneuvering motion modeled by the above models is uncoupled across $\mathsf{x}$, $\mathsf{y}$, and $\mathsf{z}$ directions. In this case, the covariance of the noise term in (13) is given by

$$\begin{aligned} &\mathrm{cov}(Gw_k) = \mathrm{diag}[\mathrm{var}(w_x)Q_2, \mathrm{var}(w_y)Q_2, \mathrm{var}(w_z)] \\ &Q_2 = \begin{bmatrix} T^4/4 & T^3/2 \\ T^3/2 & T^2 \end{bmatrix}. \end{aligned} \tag{15}$$

This model is defined directly in discrete time and is not entirely equivalent to the above continuous-time model.

The discrete-time equivalent of the above continuous-time model is [2]

$$x_{k+1} = \mathrm{diag}[F_2, F_2, 1]x_k + w_k \tag{16}$$

[4]Note that velocity is a vector and speed is its magnitude.

[5]In this survey, the regular symbol $x$ and $z$ are used for simplicity to denote the state and measurement vectors, respectively, while the positions along the $\mathsf{x}$, $\mathsf{y}$, and $\mathsf{z}$ axes are denoted by sans serif font $(\mathsf{x}, \mathsf{y}, \mathsf{z})$.

[6]For convenience, we use the shorthand notation $A = \mathrm{diag}[A_1, A_2, \ldots, A_n]$ to denote a block-diagonal matrix $A$, where $A_i$ and $A$ are not necessarily square matrices.

where

$$\text{cov}(w_k) = \text{diag}\left[\frac{S_x}{T}\tilde{Q}_2, \frac{S_y}{T}\tilde{Q}_2, \frac{S_z}{T}\right]$$
$$\tilde{Q}_2 = \begin{bmatrix} T^4/3 & T^3/2 \\ T^3/2 & T^2 \end{bmatrix}. \tag{17}$$

Note the difference between the discrete-time equivalent (16) and the direct discrete-time counterpart (13).

In a 2D scenario where the altitude $\mathsf{z}$ is not considered, the above models take the more popular form, respectively,

$$\begin{aligned} \dot{x}(t) &= A_{cv}x(t) + B_{cv}w(t) \\ x_{k+1} &= F_{cv}x_k + G_{cv}w_k \\ x_{k+1} &= F_{cv}x_k + w_k. \end{aligned} \tag{18}$$

The above models (11), (13), (16), and (18) are known as the continuous- and discrete-time *constant-velocity* (CV) models, or more precisely, "nearly-constant-velocity models." Equation (18) is in fact a "(small) *white acceleration model*" since accelerations along $\mathsf{x}$ and $\mathsf{y}$ directions are modeled as (small) white noise. The term "(nearly) constant-velocity model" emphasizes that these accelerations are small. Note that the control input $u$ is zero in the nonmaneuver models, although in reality an actual thrust of the target may be present to balance other forces so as to maintain the motion. Also, inclusion of any unnecessary component (e.g., acceleration) in the state vector would degrade tracking performance.

## IV. COORDINATE-UNCOUPLED MANEUVER MODELS

The control input $u$ responsible for a target maneuver is primarily deterministic in nature and most often unknown to the tracker. A natural way is to model it as an *unknown, deterministic* process and estimate this process from measurement data during tracking. Such *deterministic input models* are the basis for the so-called input estimation method (see, e.g., [16, 18–25]). Due to a lack of knowledge of its dynamics, this unknown process is often assumed to be piecewise constant and treated as an unknown time-invariant parameter over a time window. The main difficulty then lies in the determination of the input level and the instants at which the input jumps. This method is covered in detail in a subsequent part of this survey, of which [16] is a preliminary version.

An alternative is to model the input $u$ as a random process, which is in fact much more popular than the above deterministic modeling. Models in this class proposed in the literature can be largely classified into three groups as follows.

1) White noise models: The control input is modeled as white noise. This includes constant-velocity, constant-acceleration, and polynomial models.

2) Markov process models: The control input is modeled as a Markov process, which has a time autocorrelation. This includes the well-known Singer model, its various extensions, and some other models.

3) Semi-Markov jump process models: The control input is modeled as a semi-Markov jump process.

Most target maneuvers are coupled across different coordinates. For simplicity, however, many maneuver models developed assume that this coordinate coupling is weak and can be neglected. This is particularly the case for those that model the control input $u$ as a random process. As a consequence, we need to consider only a generic coordinate direction.

Let $\mathsf{x}$, $\dot{\mathsf{x}}$, and $\ddot{\mathsf{x}}$ be the target position, velocity, and acceleration along a *generic* direction, respectively. Specifically,

$$\ddot{\mathsf{x}}(t) = a(t). \tag{19}$$

The models discussed in this section differ in how the function $a(t)$ is defined.

In this section, the state vector is always taken to be $x = [\mathsf{x}, \dot{\mathsf{x}}, \ddot{\mathsf{x}}]'$ along the generic direction, unless stated otherwise explicitly.

### A. White-Noise Acceleration Model

The simplest model for a target maneuver is the so-called *white-noise acceleration model* [2]. It assumes that the target acceleration $\ddot{\mathsf{x}}(t)$ is an independent process (strictly white noise). It differs from the nonmaneuver model of Section III only in the noise level: the white noise process $w$ used to model the effect of the control input $u$ has a much higher intensity than the one used in a nonmaneuver model. A maneuver by its very nature aims at accomplishing a certain task and thus is rarely independent with respect to time. The main attractive feature of this model is its simplicity. It is sometimes used when the maneuver is quite small or random. It is also used in some maneuvering target tracking techniques, such as the so-called noise-level adjustment, discussed in a subsequent part of this survey (see [16] for a preliminary version).

### B. Wiener-Process Acceleration Model

The second simplest model is the so-called *Wiener-process acceleration model* [2]. It assumes that the acceleration is a Wiener process, or more generally and precisely, the acceleration is a process with independent increments, which is not necessarily a Wiener process. It is also referred to simply as the *constant-acceleration* (CA) model or more precisely "nearly-constant-acceleration model."

This model has two commonly used versions. The first one, referred to as the *white-noise jerk model*, assumes that the acceleration *derivative* (i.e., "jerk") $\dot{a}(t)$ is an independent process (white noise) $w(t)$: $\dot{a}(t) = w(t)$, with power spectral density $S_w$. The corresponding state-space representation is $\dot{x}(t) = A_3 x(t) + B_3 w(t)$, where

$$A_3 = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0 \end{bmatrix}, \qquad B_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}. \tag{20}$$

Its discrete-time equivalent is

$$x_{k+1} = F_3 x_k + w_k, \qquad F_3 = \begin{bmatrix} 1 & T & T^2/2 \\ 0 & 1 & T \\ 0 & 0 & 1 \end{bmatrix} \tag{21}$$

where

$$Q = \mathrm{cov}(w_k) = S_w Q_3, \qquad Q_3 = \begin{bmatrix} T^5/20 & T^4/8 & T^3/6 \\ T^4/8 & T^3/3 & T^2/2 \\ T^3/6 & T^2/2 & T \end{bmatrix}. \tag{22}$$

Note that $S_w$ is the power spectral density, not the variance, of the continuous-time white noise $w(t)$.

The second version can be called *Wiener-sequence acceleration model*. It assumes that the acceleration increment is an independent (white noise) process. An acceleration increment over a time period is the integral of the jerk over the period. This model is most conveniently expressed in discrete time directly, given by

$$x_{k+1} = F_3 x_k + G_3 w_k, \qquad G_3 = \begin{bmatrix} T^2/2 \\ T \\ 1 \end{bmatrix}. \tag{23}$$

Note that its noise term has a covariance different from that of the white-noise jerk model:

$$Q = \mathrm{cov}(G_3 w_k) = \mathrm{var}(w_k) \begin{bmatrix} T^4/4 & T^3/2 & T^2/2 \\ T^3/2 & T^2/2 & T \\ T^2/2 & T & 1 \end{bmatrix}. \tag{24}$$

The above models are simple but crude. Actual maneuvers seldom have (nearly) constant accelerations that are uncoupled across coordinate directions.

As explained before, a continuous-time model is more accurate than its discrete-time versions for most practical situations since a target moves continuously over time. The assumption of the direct discrete-time CA model (i.e., the second version above) that the acceleration increment $\Delta a_k = a_{k+1} - a_k = a(t_{k+1}) - a(t_k)$ is independent across different sampling intervals is hardly justifiable, except for its simplicity and mathematical tractability. Were this assumption true for a sampling period $T$, it would not be true in general for any other sampling period $T'$ unless it is a multiple of $T$: $T' = nT$. Even if such periods exist, we would not be so lucky that one of them is used by chance.

## C. Polynomial Models

It is well known that any continuous target trajectory can be approximated by a polynomial of a certain degree to an arbitrary accuracy. As such, it is possible to model target motion by an $n$th-degree polynomial [2] in the Cartesian coordinates:

$$\begin{bmatrix} \mathsf{x}(t) \\ \mathsf{y}(t) \\ \mathsf{z}(t) \end{bmatrix} = \begin{bmatrix} a_0 & a_1 & \cdots & a_n \\ b_0 & b_1 & \cdots & b_n \\ c_0 & c_1 & \cdots & c_n \end{bmatrix} \begin{bmatrix} 1 \\ t \\ \vdots \\ t^n/n! \end{bmatrix} + \begin{bmatrix} w_x(t) \\ w_y(t) \\ w_z(t) \end{bmatrix} \tag{25}$$

with a certain choice of the coefficients $a_i, b_i, c_i$, where $(\mathsf{x},\mathsf{y},\mathsf{z})$ are the position coordinates and $(w_x, w_y, w_z)$ are the corresponding noise terms. Such an $n$th-degree polynomial model amounts to assuming the $n$th time derivative of the position is (nearly) constant (i.e., the position deviation from such a constant $n$th derivative motion is equal to the noise $w$). The CV and CA models described above are special cases (for $n = 1, 2$, respectively) of this general $n$th-degree model with white noise $w(t)$. Note that this model is coordinate uncoupled if $w_x, w_y, w_z$ are uncorrelated. Also, an $n$th-degree polynomial has $(n+1)$ parameters per coordinate. That is why a model of an $n$th-degree polynomial is often called an $(n+1)$th-order model.

This model in its general setting does not appear very attractive for tracking for several reasons. Such models are usually good for fitting to a set of data, that is, for smoothing problem; however, the primary purpose of tracking is prediction and filtering, rather than fitting or smoothing. It is difficult to develop an uncomplicated and efficient method to determine the coefficients $a_i, b_i, c_i$ systematically in a general setting. Nevertheless, many special polynomial models have been developed for target tracking. In fact, most of the models discussed in this section can be viewed as special cases of this general polynomial model with different models for the noise $w(t)$.

## D. Singer Acceleration Model—Zero-Mean First-Order Markov Model

In stochastic modeling, a random variable is used to represent an unknown time-invariant quantity, while an unknown time-varying quantity is modeled by a random process. As far as temporal properties are concerned, white noise constitutes the simplest class of random processes. The second simplest class is either the processes with independent increments, represented by the Wiener processes, or the so-called Markov processes, which include the Wiener processes and white noise as special cases.

White noise is "isolated" in time since its value at one time is uncoupled of any other time, while a Markov process is "local" in time because its value at one time depends on values at other times only through its nearest neighbors. Consequently, it is natural to consider a Markov process model whenever white noise models are not good enough.

The *Singer model* [26] assumes that the target acceleration $a(t)$ is a zero-mean first-order stationary Markov process with autocorrelation $R_a(\tau) = E[a(t+\tau)a(t)] = \sigma^2 e^{-\alpha|\tau|}$, or equivalently, power spectrum $S_a(\omega) = 2\alpha\sigma^2/(\omega^2+\alpha^2)$. Such a process $a(t)$ is the state process of a linear time-invariant system[7]

$$\dot{a}(t) = -\alpha a(t) + w(t), \qquad \alpha > 0 \tag{26}$$

where $w(t)$ is zero-mean white noise with constant power spectral density $S_w = 2\alpha\sigma^2$. Its discrete-time equivalent is

$$a_{k+1} = \beta a_k + w_k^a \tag{27}$$

where $w_k^a$ is a zero-mean white noise sequence with variance $\sigma^2(1-\beta^2)$ and $\beta = e^{-\alpha T}$. The state-space representation of the continuous-time Singer model is

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\alpha \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w(t). \tag{28}$$

Its discrete-time equivalent is

$$x_{k+1} = F_\alpha x_k + w_k = \begin{bmatrix} 1 & T & (\alpha T - 1 + e^{-\alpha T})/\alpha^2 \\ 0 & 1 & (1-e^{-\alpha T})/\alpha \\ 0 & 0 & e^{-\alpha T} \end{bmatrix} x_k + w_k. \tag{29}$$

The exact covariance of $w_k$ is a function of $\alpha$ and $T$ and can be found in, e.g., [26, 1, 2].

The success of the Singer model relies on an accurate determination of the parameters $\alpha$ and $\sigma^2$ [27]. The parameter $\alpha = 1/\tau$ is the reciprocal of the maneuver time constant $\tau$ and thus depends on how long the maneuver lasts. For example for an aircraft, $\tau \approx 60$ s for a lazy turn and $\tau \approx 10$–$20$ s for an evasive maneuver, as suggested in [26]. The parameter $\sigma^2 = E[a(t)^2]$ is the "instantaneous variance" of the acceleration. It was proposed in [26] to model the distribution of the acceleration by the following *ternary-uniform mixture* (see Fig. 1): the target may move without acceleration with probability $P_0$; accelerate or decelerate at a maximum rate $a_{\max}$ with equal probability $P_{\max}$; or accelerate or decelerate at a rate uniformly distributed over $(-a_{\max}, a_{\max})$. It turns out that

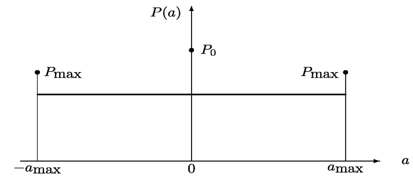

Fig. 1. Ternary-uniform mixture pdf.

$$\sigma^2 = \frac{a_{\max}^2}{3}(1 + 4P_{\max} - P_0)$$

where $P_{\max}$, $P_0$, and $a_{\max}$ are design parameters. We emphasize that Singer model is a *maneuver* model and thus $P_0$ should be probability of having a zero acceleration *during a maneuver*, rather than probability of a nonmaneuvering motion. Note also that this ternary-uniform mixture distribution of acceleration can obviously be used for other maneuver models and it is used here only to determine $\sigma^2$.

It is clear from (28)–(29) that in the limit:

1) As the maneuver time constant $\tau$ increases (i.e., $\alpha T$ decreases), the Singer model reduces to the CA model [more precisely, to the white-noise jerk model since $\mathrm{cov}(w_k)$ reduces to $Q$ of (22) instead of $Q$ of (24)]. If the following direct discrete-time counterpart of (28) were set up

$$x_{k+1} = \begin{bmatrix} 1 & T & T^2/2 \\ 0 & 1 & T \\ 0 & 0 & \beta \end{bmatrix} x_k + G_3 w_k \tag{30}$$

then its limit as $\tau$ increases would be the Wiener-sequence acceleration model. This relationship between the Singer and CA models makes sense since the deterministic part of the acceleration in the Singer model becomes constant in the limit as $\tau$ increases.

2) On the other hand, as the maneuver time constant $\tau$ decreases (i.e., $\alpha T$ increases), the Singer model reduces to the CV model. In this case, the acceleration becomes white noise.

Consequently, for a choice of $0 < \alpha T < \infty$, the Singer model corresponds to a motion in between of (nearly) constant velocity and (nearly) constant acceleration. It should thus be clear that the Singer model has wider coverage than the CV and CA models.

Many other models have been proposed (see, e.g., [28–33]) which are equivalent to or are simple variants of the Singer model. It has been applied in [33] for angular acceleration as well as linear acceleration. It is interesting that the same model, except that $\alpha$ is data-dependent and time-varying, was obtained approximately in [34] for the angular velocity components (pitch and yaw) of the relative motion between the target and seeker based on well-known relationships in classical mechanics.

[7] A stationary Markov process with a rational power spectrum (as is the case here) is equivalent to the state of an asymptotically stable linear time-invariant system excited by strictly white noise: Every such process can be represented as the state of such a system and the state of such a system is such a Markov process.

The Singer acceleration model is a popular model for target maneuvers (see, e.g., [35–40], [1, 2]). It was the first model that characterizes the unknown target acceleration as a time-correlated (i.e., colored) stochastic process, and has served as a basis for further development of many effective maneuver models.

The Singer model is in essence an a priori model since it does not use online information about the target maneuver, although it can be made adaptive through an adaptation of its parameters $\alpha$, $P_{\max}$, $P_0$, $T$, and/or $a_{\max}$. We cannot reasonably expect any a priori model to have a remarkable effectiveness for the diverse acceleration situations of actual target maneuvers. As a consequence of its a priori nature, the Singer model is also symmetric in that the assumed ternary-uniform mixture distribution of the acceleration is symmetric. One of the main shortcomings of the Singer model stems from this symmetry; that is, the target acceleration has zero mean at any moment. Indeed, this is almost the best one can do a priori without online information about the target maneuver. Otherwise towards where should the mean be? However, nothing really prevents us from using online information if we can withstand a little more sophistication. Several more sophisticated acceleration models have been proposed to remedy this shortcoming. They are described next.

### E. Mean-Adaptive Acceleration Model

An acceleration model, called the *"current" model* by its authors, proposed in [41], is in essence a Singer model with an adaptive mean; that is, a Singer model modified to have a non-zero mean of the acceleration: $a(t) = \tilde{a}(t) + \bar{a}(t)$, where $\tilde{a}(t)$ is the zero-mean Singer acceleration process, defined by (26), and $\bar{a}(t)$ is the mean of the acceleration, artificially assumed constant over each sampling interval. Such a non-zero-mean acceleration satisfies

$$\dot{a}(t) = -\alpha\tilde{a}(t) + w(t) \qquad \text{or} \qquad \dot{a}(t) = -\alpha a(t) + \alpha\bar{a}(t) + w(t) \tag{31}$$

since $\dot{a}(t) = \dot{\tilde{a}}(t)$ over any sampling interval. The estimate $\hat{a}_k$ of $a_k$ from all available online information (i.e., the sequence $z^k$ of observations through time $k$) is taken to be the "current" value of the mean $\bar{a}_{k+1}$, hence, the name. It is potentially more effective than the Singer model.

The state-space representation of this model is

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & -\alpha \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ \alpha \end{bmatrix} \bar{a}(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w(t) \tag{32}$$

excluding the time instants at which samples are taken since $\bar{a}(t)$ is assumed piecewise constant. The discrete-time equivalent is

$$x_{k+1} = F_\alpha x_k + \left( \begin{bmatrix} T^2/2 \\ T \\ 1 \end{bmatrix} - \begin{bmatrix} (\alpha T - 1 + e^{-\alpha T})/\alpha^2 \\ (1 - e^{-\alpha T})/\alpha \\ e^{-\alpha T} \end{bmatrix} \right) \bar{a}_k + w_k \tag{33}$$

where $F_\alpha$ was given in (29). These two equations differ from the Singer model only in the additional terms associated with $\bar{a}(t)$ and $\bar{a}_k$, respectively. For example, the noise $w(t)$ and $w_k$ are identical to those in the Singer model. Note also that this model corresponds to a Singer model with noise $w$ of a non-zero adaptive mean.

A key underlying assumption of the "current" model in [41] (but not so stated explicitly) is that $\bar{a}_{k+1} = \hat{a}_k$, or more specifically,

$$\bar{a}_{k+1} \triangleq E[a_{k+1} \mid z^k] = E[a_k \mid z^k] \triangleq \hat{a}_k \tag{34}$$

where $z^k$ stands for all measurements through time $t_k$. This is questionable and can actually be avoided. Since the last equation of (33) reads

$$a_{k+1} = e^{-\alpha T} a_k + (1 - e^{-\alpha T})\bar{a}_k + w_k^a \tag{35}$$

we propose to improve the "current" model by replacing $\bar{a}_{k+1} = \hat{a}_k$ with the following recursion

$$\begin{aligned} \bar{a}_{k+1} &= E[a_{k+1} \mid z^k] = e^{-\alpha T} E[a_k \mid z^k] + (1 - e^{-\alpha T})\bar{a}_k \\ &= e^{-\alpha T}\hat{a}_k + (1 - e^{-\alpha T})\bar{a}_k. \end{aligned} \tag{36}$$

Such a relationship makes better sense because $\bar{a}_{k+1}$ depends on the current estimate $\hat{a}_k$ as well as past information $\bar{a}_k$. However, what is resulting is no longer a purely *current* model.

In the "current" model, the a priori (unconditional) probability density $f(a_{k+1})$ of the acceleration $a_{k+1}$ at time $k+1$ in the Singer model is replaced by a conditional density $f(a_{k+1} \mid \hat{a}_k)$. Clearly, this conditional density carries more accurate information and is better to be used than the a priori density, as explained before. The following *conditional* Rayleigh density (see Fig. 2) was proposed in [41] for $a_{k+1}$:

$$f(a \mid \hat{a}_k) = \begin{cases} c_k^{-2}(a_{\max} - a)\exp\left(\dfrac{-(a_{\max} - a)^2}{2c_k^2}\right)\mathbf{1}(a_{\max} - a) & \hat{a}_k > 0 \\ c_k^{-2}(a - a_{-\max})\exp\left(\dfrac{-(a - a_{-\max})^2}{2c_k^2}\right)\mathbf{1}(a - a_{-\max}) & \hat{a}_k < 0 \end{cases}$$

where $\mathbf{1}(\cdot)$ is the unit step function; $a_{-\max}$ is the negative acceleration limit, not necessarily equal to $-a_{\max}$; and $c_k$ is an $\hat{a}_k$-dependent parameter. Note that the ternary-uniform distribution assumption in the Singer model was made only out of the need to calculate the error variance of the acceleration. Similarly, this conditional Rayleigh assumption was

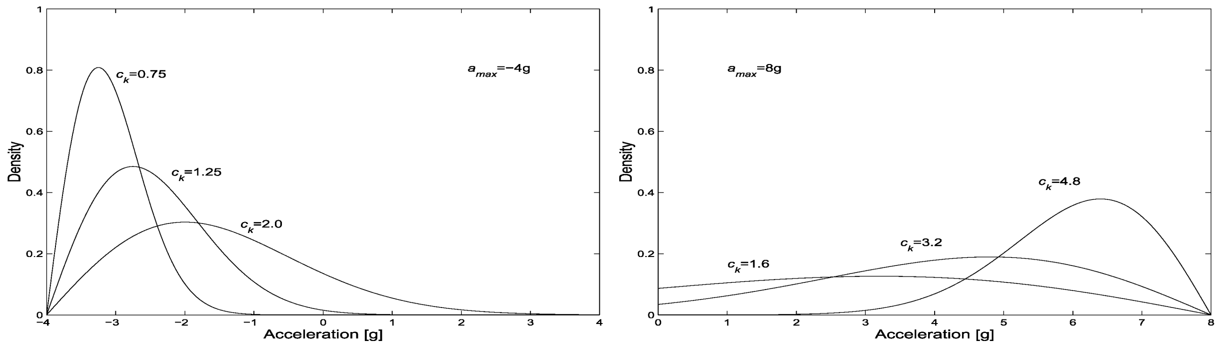


Fig. 2. Conditional Rayleigh model.

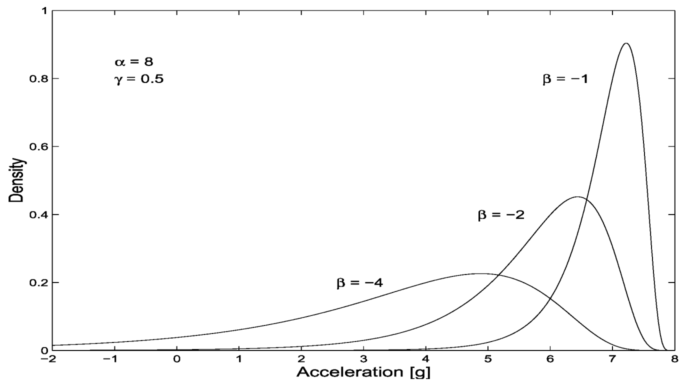


Fig. 3. Asymmetric pdfs of normal acceleration.

made for the sole purpose of obtaining the variance of the acceleration prediction, which turns out to be

$$\sigma_k^2 \triangleq E[(a_{k+1} - \bar{a}_{k+1})^2 \mid \hat{a}_k] = \begin{cases} \dfrac{4-\pi}{\pi}(a_{\max} - \hat{a}_k)^2 & \hat{a}_k > 0 \\ \dfrac{4-\pi}{\pi}(a_{-\max} + \hat{a}_k)^2 & \hat{a}_k < 0 \end{cases}.$$

This result is valid only under the simplifying assumption that $E[a_{k+1} \mid z^k] = E[a_{k+1} \mid \hat{a}_k]$, which is the case when $\hat{a}_k$ is a sufficient statistic of $z^k$ for $a_{k+1}$.

Applications of the "current" model can be found in the literature, e.g., to a benchmark tracking problem [42].

## F. Asymmetrically Distributed Normal Acceleration Model

Target acceleration can be decomposed along two directions: lift (normal to the target velocity and wing directions for an aircraft) and thrust or drag (along the velocity direction). Each component can be modeled by a time-correlated random process. The nonnormal component can be modeled well by the Singer model. However, this is often not the case for lift, which is usually the dominant one, especially during a maneuver. Its direction is determined by the target aspect angle and its magnitude can be modeled as a colored random process with an asymmetrical distribution. It was proposed in [43] that the normal acceleration $a_n(t)$ be modeled as an asymmetric and deterministic function of a zero-mean first-order Gauss-Markov process $b(t)$:

$$a_n(t) = \alpha + \beta e^{\gamma b(t)} \tag{37}$$

where $\alpha, \beta, \gamma$ are design parameters, depending on the particular target type; $b(t)$ satisfies the equation of the Single model $\dot{b}(t) = -(1/\tau)b(t) + w(t)$, where $\tau$ is the correlation time constant of $b(t)$, which in general differs from that of $a_n(t)$. As shown in Fig. 3, three typical choices of $\alpha, \beta, \gamma$ and the corresponding (highly asymmetrical) probability density functions (pdfs) were given in [43], including one $[(\alpha, \beta, \gamma) = (8, -4, 0.5)]$ that is considered typical of modern piloted aircraft in evasive maneuvers.

Both pros and cons of this model stem from the fact that the parameters $\alpha, \beta, \gamma$ are target-type specific. It is more accurate than the Singer model at the cost of designing these parameters, which requires knowledge of the target type, obtained either a priori or a posteriori. The shortcoming of the Singer model with a symmetric pdf is overcome by the use of additional information of target type in this model.

Note that the temporal correlation of the acceleration in this model does not have a rational power spectrum and is much more complicated than in the Singer model. It would be interesting to compare this model with a Singer model of the normal acceleration $a_n$ having the initial asymmetric pdf $a_n(t_0) = \alpha + \beta e^{\gamma b(t_0)}$ with the same parameters $\alpha, \beta, \gamma$ as above.

## G. Markov Models for Oscillatory Targets

It is not uncommon in reality that acceleration along one coordinate direction is oscillatory due to, e.g., wind sway or platform roll. The Singer model is not very suitable for such practical maneuvers. The autocorrelation of such acceleration may be described by

$$\begin{aligned} R_a(\tau) &= \sigma_a^2 e^{-\alpha|\tau|}\cos(\omega_c\tau) \\ &= \sigma_a^2 e^{-\zeta\omega_n|\tau|}\cos(\omega_n\sqrt{1-\zeta^2}\tau) \\ \omega_n^2 &= \alpha^2+\omega_c^2, \zeta = \alpha/\omega_n \end{aligned} \tag{38}$$

where $\sigma_a^2$, $\alpha$, $\omega_c$, $\zeta$, and $\omega_n$ are average power, damping coefficient, actual (damped) frequency, damping ratio, and undamped natural frequency of the target acceleration, respectively.

Such an acceleration process is the response of a *second-order prewhitening system* to zero-mean white noise input $w(t)$ with power spectral density $2\alpha\sigma_a^2$. It has transfer function

$$H(s) = \frac{s+\omega_n}{s^2+2\zeta\omega_n s+\omega_n^2} \tag{39}$$

and is described by (see, e.g., [44, 12, 11])

$$\begin{bmatrix}\dot a(t)\\ \dot d(t)\end{bmatrix} = \begin{bmatrix}0 & 1\\ -\omega_n^2 & -2\zeta\omega_n\end{bmatrix}\begin{bmatrix}a(t)\\ d(t)\end{bmatrix} + \begin{bmatrix}1\\ (1-2\zeta)\omega_n\end{bmatrix} w(t) \tag{40}$$

where $d(t) = \dot a(t) - w(t)$, which can be called acceleration drift. The state-space model for $x = [\mathsf{x}, \dot{\mathsf{x}}, \ddot{\mathsf{x}}, d]'$ is

$$\dot x(t) = Ax(t) + Bw(t) = \begin{bmatrix}0 & 1 & 0 & 0\\ 0 & 0 & 1 & 0\\ 0 & 0 & 0 & 1\\ 0 & 0 & -\omega_n^2 & -2\zeta\omega_n\end{bmatrix} x(t) + \begin{bmatrix}0\\ 0\\ 1\\ (1-2\zeta)\omega_n\end{bmatrix} w(t). \tag{41}$$

Note that the autocorrelation $R_a(\tau)$ is not periodic anymore if $\zeta \geq 1$. More generally, if the target acceleration has autocorrelation

$$R_a(\tau) = \frac{\sigma_a^2}{\cos\theta} e^{-\zeta\omega_n|\tau|}\cos(\omega_n\sqrt{1-\zeta^2}|\tau| - \theta) \tag{42}$$

then the state-space model [44, 12, 11] is given by (41) with $Bw(t)$ replaced by $[0,0,b,c]'w(t)$, where

$$\begin{aligned} b &= \frac{2\sigma_a^2}{\cos\theta}\omega_n \sin(\varphi-\theta) \\ c &= \frac{2\sigma_a^2}{\cos\theta}\omega_n^3 \sin(\varphi+\theta) \\ \varphi &= \tan^{-1}\frac{\zeta}{\sqrt{1-\zeta^2}} \end{aligned} \tag{43}$$

and the corresponding transfer function of the prewhitening system is

$$H(s) = \frac{bs+c}{s^2+2\zeta\omega_n s+\omega_n^2}. \tag{44}$$

This model was utilized in a multiple-model tracker in [45].

The discrete-time equivalent of model (41) is found to be

$$x_{k+1} = \begin{bmatrix}1 & T & F_{13} & F_{14}\\ 0 & & & \\ 0 & & F(\omega_c,\alpha) & \\ 0 & & & \end{bmatrix} x_k + w_k \tag{45}$$

where

$$F(\omega_c,\alpha) = \begin{bmatrix}1 & \dfrac{2\alpha\omega_c - e^{-\alpha T}(2\alpha\omega_c\cos\omega_c T + (\alpha^2-\omega_c^2)\sin\omega_c T)}{\omega_c(\alpha^2+\omega_c^2)} & \dfrac{\omega_c - e^{-\alpha T}(\omega_c\cos\omega_c T + \alpha\sin\omega_c T)}{\omega_c(\alpha^2+\omega_c^2)}\\ 0 & \dfrac{e^{-\alpha T}(\omega_c\cos\omega_c T + \alpha\sin\omega_c T)}{\omega_c} & \dfrac{e^{-\alpha T}\sin\omega_c T}{\omega_c}\\ 0 & -\dfrac{(\alpha^2+\omega_c^2)e^{-\alpha T}\sin\omega_c T}{\omega_c} & \frac{e^{-\alpha T}(\omega_c\cos\omega_c T - \alpha\sin\omega_c T)}{\omega_c}\end{bmatrix} \tag{46}$$

$$F_{13} = \frac{\omega_c(-3\alpha^2+\omega_c^2+2\alpha^3 T+2\alpha\omega_c^2 T) - e^{-\alpha T}[\omega_c(-3\alpha^2+\omega_c^2)\cos\omega_c T - \alpha(\alpha^2-3\omega_c^2)\sin\omega_c T]}{\omega_c(\alpha^2+\omega_c^2)^2}$$

$$F_{14} = \frac{\omega_c(-2\alpha+\alpha^2 T+\omega_c^2 T) + e^{-\alpha T}[2\alpha\omega_c\cos\omega_c T + (\alpha^2-\omega_c^2)\sin\omega_c T]}{\omega_c(\alpha^2+\omega_c^2)^2}$$

and $\text{cov}(w_k) = S_w\int_0^T e^{A\tau}BB'(e^{A\tau})'d\tau$ can be found (e.g., by Mathematica) but is too tedious to be given here.

A similar but slightly more sophisticated Markov model was used in [11, 46] for a target trajectory with oscillations due to wind-induced bending. Assume that the bending is modeled by a second-order Markov system with undamped natural frequency $\omega_n$ and damping ratio $\zeta$, along with a standard model of winds (i.e., a Singer model with correlation time constant $\tau = 1/\mu$), such that the prewhitening system has the transfer function

$$H(s) = \frac{S_w^{1/2} s^2}{(s+\mu)(s^2 + 2\zeta\omega_n s + \omega_n^2)} = \frac{S_w^{1/2} s^2}{s^3 + \alpha s^2 + \beta s + \gamma} \tag{47}$$

where $\alpha = \mu + 2\zeta\omega_n$, $\beta = \omega_n^2 + 2\mu\zeta\omega_n$, and $\gamma = \mu\omega_n^2$. This system is described by [11, 46]

$$\begin{bmatrix} \dot{a} \\ \ddot{a} \\ \dddot{a} \end{bmatrix}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ -\gamma & -\beta & -\alpha \end{bmatrix} \begin{bmatrix} a \\ \dot{a} \\ \ddot{a} \end{bmatrix} + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w(t) \tag{48}$$

where $w(t)$ has power spectrum $S_w = c_3\sigma^2$ and $\sigma^2 = R(0)$ is the mean-square value of the target trajectory with the following autocorrelation

$$R(\tau) = c_1 e^{-\mu|\tau|} + c_2 e^{-\zeta\omega_n|\tau|}\cos(\omega_n\sqrt{1-\zeta^2}|\tau| - \theta) \tag{49}$$

which is a combination of a first-order term and a second-order term. Here $c_1$, $c_2$, $c_3$, and $\theta$ are constants, depending on $\alpha$, $\beta$, $\gamma$, and $\sigma$ [46]. Its discrete-time equivalent can be found in [46].

### H. Markov Acceleration Model for Constant Turns

A typical target maneuver, such as a turn, often has an approximately constant speed and turn rate. The original Singer model is not very good for such motion, referred to as a *constant turn* in this survey. Let the acceleration $a$ along the $\mathsf{x}$ and $\mathsf{y}$ directions of the Cartesian coordinates be $a_x$ and $a_y$, respectively: $a = [a_x, a_y]'$. Denote by $v$ the constant speed, $\phi(t)$ the (velocity) heading angle, and $\omega = \dot{\phi}$ the constant turn rate. Under the assumption that both $\phi$ and $\omega$ as random variables have symmetric distributions and are mutually independent, it can be easily shown that the zero-mean processes $a_x(t)$ and $a_y(t)$ are uncorrelated [47]. Note, however, that while the symmetry assumption is fairly reasonable in practice for a priori models, the independence assumption is questionable since the turn rate $\omega$ and the heading $\phi$ are actually related by $\omega = \dot{\phi}$ and thus dependent.

Only the model for $a_x(t)$ is considered below since the model for $a_y(t)$ can be obtained similarly. It can be easily shown from the constant-turn equation $a_x = \ddot{\mathsf{x}} = -\omega\dot{\mathsf{y}} = -\omega v \sin\phi$ and $\phi(t+\tau) = \phi(t) + \omega\tau$ that the autocorrelation of $a_x(t)$ is given by

$$\begin{aligned} R(t+\tau, t) &= \tfrac{1}{2} v^2 E[\omega^2\{\cos\omega\tau[1 - \cos 2\phi(t)] + \sin\omega\tau \sin 2\phi(t)\}] e^{-\alpha|\tau|} \\ &= \sigma^2(t,\tau) e^{-\alpha|\tau|} \end{aligned} \tag{50}$$

assuming that the magnitude of the total acceleration $a(t)$ obeys the Singer model. Clearly, $a_x(t)$ is nonstationary because its second-order statistics $R(t+\tau, t)$ depend on $t$ as well as $\tau$. The major difference between this model and the Singer model is that here $\sigma^2(t,\tau)$ is a function of $t$ and $\tau$. Simply put, this model is a modified Singer model for a typical motion, referred to as constant turn (with constant speed and turn rate). Its nonstationarity is a consequence of the constant-turn constraint. It is possible to be more accurate than the Singer model for a constant turn. The price paid is that the precise state-space form of this model is complicated. To have a time-invariant model of $a_x(t)$, it is necessary to consider only those distributions of $\phi$ for which $R(t+\tau, t) = R(\tau)$. The uniform distribution of $\phi$ over $(-\pi, \pi]$ is one of such distributions. To obtain a state-space model of $a_x(t)$, its power spectrum is derived, which is, however, not of a rational form and is dependent on the distribution of $\omega$. In [47], it was assumed that the constant turn rate is uniformly distributed over a given interval $[-\omega_{\max}, \omega_{\max}]$ and the heading angle is uniformly distributed over $(-\pi, \pi]$, and they are independent.[8] It was proposed in [47] to approximate the irrational power spectrum of $a_x(t)$ by an $n$th-order rational spectrum to obtain a simplified state-space model. This yields an $n$th-order Markov model. Such a rational approximation may be obtained numerically, as in [47]. For instance, if the power spectrum of $a_x(t)$ has the following general second-order rational approximation

$$S(s) = H(s)H(-s) \qquad \text{with} \quad H(s) = \frac{\beta_1 s + \beta_2}{s^2 + \alpha_1 s + \alpha_2} \tag{51}$$

then the causal and stable prewhitening system is given by $H(s)$. The corresponding state-space model is,[9] for $x = [\mathsf{x}, \dot{\mathsf{x}}, \ddot{\mathsf{x}}, d_x]'$,

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & -\alpha_2 & -\alpha_1 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ \beta_1 \\ \beta_2 - \alpha_1\beta_1 \end{bmatrix} w_x(t) \tag{52}$$

where $w_x(t)$ is white noise with unity power spectral density. Note that a horizontal constant turn is covered

[8]The assumptions made in [47] for this model are not consistent with each other. For example, $\phi(t+\tau) = \phi(t) + \omega\tau$ indicates that if $\phi(t) \sim \mathcal{U}[-\pi, \pi]$, $\omega \sim \mathcal{U}[-\omega_{\max}, \omega_{\max}]$ and they are independent, then $\phi(t+\tau)$ has a trapezoidal distribution rather than a uniform distribution. Note that all simplifying assumptions are actually incorrect. However, such incorrect assumptions may still be used in complex situations if it leads to great simplicity, such as that of the PDAF for approximating a Gaussian mixture by a single Gaussian. What is important is to exercise care, be reasonable, and to judge by the final outcomes.

[9]The model given here appears simpler and more reasonable than what was given in [47].

by the use of two uncoupled 4-dimensional models here, but actually it has a more accurate single 5-dimensional model (see Section V).

The second-order model given by (40) is a special case of this general second-order model (52) with $\beta_1 = 1$, $\beta_2 = \omega_n$, $\alpha_1 = 2\zeta\omega_n$, $\alpha_2 = \omega_n^2$. It follows from (44) that if $\beta_1 = b$, $\beta_2 = c$, $\alpha_1 = 2\zeta\omega_n$, $\alpha_2 = \omega_n^2$ the target acceleration of this second-order Markov model has an oscillatory yet stationary autocorrelation given by (42), but the "exact" autocorrelation is given by (50), which is nonstationary. Note also that the above uniform distribution assumption of the turn rate is better replaced by those of [48], the Singer's ternary-uniform mixture or its variants (e.g., a binary-uniform mixture or a single-point and uniform mixture) in many practical situations with additional information of the turn rate.

### I. Semi-Markov Jump Process Models

The Singer model approximates the target acceleration as a continuous-time *zero-mean* Markov process. In practice, many target maneuvers involve an acceleration of a *non-zero mean* that may be reasonably assumed piecewise constant. However, the difficulty here is that neither the time intervals over which the acceleration mean is piecewise constant nor the corresponding constant levels of the non-zero mean are known to a tracker.

One of the simplest piecewise-constant random processes, known as jump processes, is the so-called semi-Markov jump process (see [49] for an engineering-oriented description). It differs from a Markov jump process only in that it has the Markov property[10] if we consider only time instants of a jump, but not necessarily at other times, while a Markov jump process has the Markov property at all times. In other words, the past and the future states of a semi-Markov process may be coupled through the time interval it stays in a state (called *sojourn time*) as well as the present state—its value at one time may depend on values at other time instants through not only its nearest neighbor state but also the sojourn time in the state.

Several Markovian jump-mean acceleration models have been proposed. The first was the one given in [50–53]. In this method, the unknown input $u(t)$ (assumed equal to the non-zero mean of the acceleration $a$) is modeled as a finite-state semi-Markov jump process. Specifically, it was assumed that the possible *mean* values of the acceleration are quantized into $n$ known levels $\bar{a}_1,\ldots,\bar{a}_n$, and the sequence $\langle u(t_k)\rangle$ of the input among these levels is a semi-Markov process (a sojourn-time dependent Markov chain) with known transition probability $P\{u(t_k) = \bar{a}_j \mid u(t_{k-1}) = \bar{a}_i\}$, $i,j = 1,\ldots,n$, and sojourn-time probability distribution function $P_{ij}(\tau) = P\{\tau_{ij} \le \tau\}$, where $\tau_{ij} = t_k - t_{k-1}$ is the sojourn time at level $\bar{a}_i$ before it jumps to level $\bar{a}_j$. Although the concept of such a semi-Markov jump process formalism was introduced in [50, 51], only exponentially distributed sojourn time was considered therein. Note that if the sojourn time has an exponential distribution, the input process $u(t)$ is in fact Markov—the semi-Markov formulation is not needed.[11] Moreover, the following model of the acceleration $a(t)$ as a combination of the above jump-mean model and the Singer model was proposed:

$$a(t) = -\beta v(t) + u(t) + \tilde{a}(t) \tag{53}$$

where $\tilde{a}(t)$ is the Singer acceleration of (26), $v$ is the velocity, $u(t)$ is the unknown acceleration mean, and $\beta$ is a drag coefficient. The corresponding continuous-time state-space representation is, for $x = [\mathsf{x},\dot{\mathsf{x}},\ddot{\mathsf{x}}]'$,

$$\dot{x}(t) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & -\beta & 1 \\ 0 & 0 & -\alpha \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 1 \\ 0 \end{bmatrix} u(t) + \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} w(t). \tag{54}$$

In other words, the target acceleration is modeled as the Singer acceleration with non-zero mean that is intended to be a semi-Markov (but actually Markov) jump process. This model can be referred to as a *Markovian jump-mean acceleration model*. The unknown acceleration mean $u(t)$ is estimated by a weighted sum of the quantization levels: $\hat{u}(t) = \sum_{i=1}^{n} \bar{a}_i P\{u(t) = \bar{a}_i \mid z(s),\ s \le t\}$, where as in a multiple-model formulation, the weight is the posterior probability of each level being the correct one, using all online measurements $z(s), s \le t$, as well as the initial model probabilities, model transition probabilities, and the sojourn time distribution. Fig. 4 depicts some representative random processes as models of the target acceleration. Note that a semi-Markov jump process model for the acceleration is more effective than a white noise model with a mean of random jumps.

Three important issues associated with this model are the design of the input quantization levels, the transition probabilities, and the sojourn time. This is

[10] That is, the past and future states are independent given the present state.

[11] A semi-Markov process with exponentially distributed sojourn time in each state is actually a Markov process. This is a consequence of the unique memoryless property of the exponential distribution—knowledge of time already spent in a state does not alter distribution of a future state. Similarly, a discrete-time semi-Markov process is Markov only if the sojourn time has a geometric distribution.

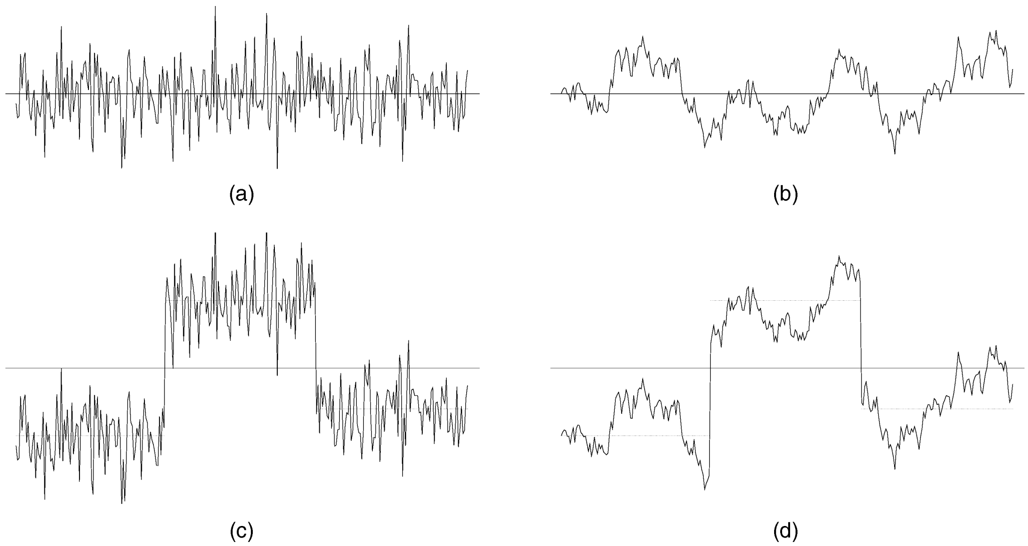


Fig. 4. Representative random acceleration process models. (a) Zero-mean white noise. (b) Zero-mean colored noise. (c) Jump-mean white noise. (d) Jump-mean colored noise.

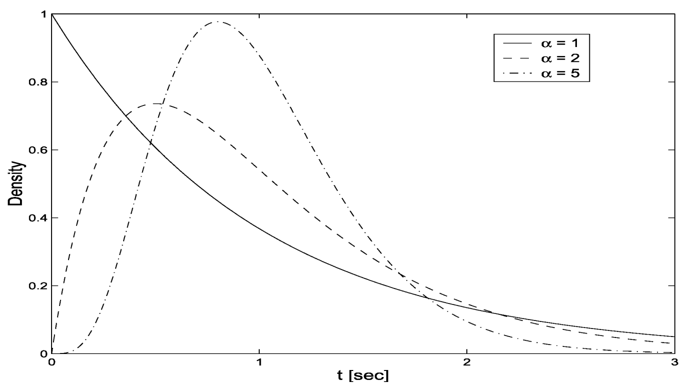


Fig. 5. Gamma densities with unity mean value.

similar to model-set design for the multiple-model method [54]. This should not be a surprise because this jump-mean model actually amounts to multiple models of the (quantized) input in a degenerated form.

The model proposed in [50, 51] for the unknown input $u$ is a discrete-time, finite-state (semi-)Markov process. A continuous-time counterpart was proposed in [55]. A sojourn-time dependent Markov chain model for target motion was proposed in [56] in the context of the multiple-model approach (see [17] also).

For simplicity, the sojourn time in a state is oft modeled as having an exponential distribution in the above "semi-Markov" formalism. The point (or counting) process associated with such an independent sojourn time process is a Poisson process. Such a sojourn time $\tau$ has a monotonically decreasing probability density. This implies that very small $\tau$ would occur most frequently of all $\tau$ within a time interval of the same length. This is not consistent with the distribution of the durations of practical target motions. To correct this deficiency, it was proposed in [57] that the sojourn time be assumed an independent process having a gamma distribution (see Fig. 5), which includes exponential as a special case with $\alpha = 1$. Such a semi-Markov process and its embedded point process are both known as *gamma-renewal processes* [58] and are generalization of the above Markov process with exponential sojourn time and its embedded Poisson process. The word "renewal" was coined in stochastic processes with regard to the failure times of certain equipment (i.e., sojourn times in our case), and thus the value of the counting process models the number of renewals that must be made to keep the equipment in working order.

Although it appears more representative of the actual maneuvers, the *renewal model* is rather complicated due to its non-Markovian nature. However, similar to modeling of the non-Markov state process of a system driven by colored noise, the prewhitening technique can be used here. The transfer function of this (usually first- or second-order) prewhitening system or its approximation can be obtained from the equilibrium power spectrum of the semi-Markov maneuver process. This leads to an (approximate) state-space representation of the renewal model. Simulation results of an application of the renewal model to an agile target that executes constant turns indicates that the performance improvement is not commensurate with the model and algorithm complexity [57], but is significant for an image-enhanced tracking system based on the fusion of a microwave radar and an infrared imaging sensor [59].

The renewal model was proposed in [57] for the turn rate. It can clearly be applied for target acceleration, its mean value, etc. In fact, following the idea of [50, 51] for an acceleration process, the turn rate can be modeled as the sum of its mean that obeys the above renewal model and a zero-mean first-order Markov process that obeys the Singer model. It would be interesting to compare its applicability with that of the renewal model above.

## J. Jerk Models

*Acceleration Models versus Jerk Models*: Jerk is the derivative of acceleration. In most coordinate-uncoupled models, it is the target acceleration that is chosen to be the descriptor of a target maneuver and modeled as a random process. This is most natural from mechanics, kinematics, and vehicle dynamics since acceleration is directly related to the force acting on the target. That is why target acceleration is usually taken to be the control input $u$. However, for some targets, particularly agile targets, it may be more convenient to use a random jerk process to model the target maneuvers. A jerk model differs from an acceleration model usually in simplicity, depending on whether the target motion is better described by a random process model of the jerk or the acceleration. On the other hand, however, jerk in a jerk model must be estimated, the accuracy of which is usually by far poorer than acceleration estimates since only position (and Doppler) measurements are usually available.

*First-Order Markov Jerk Model*: The first-order Markov model as proposed by Singer is for target acceleration. However, the same modeling method can be applied to other target functions, e.g., target jerk. This method was indeed applied in [60, 61] to the jerk process as the maneuver forcing function; that is, the jerk is modeled as a zero-mean first-order Markov process, exactly in the same manner as the Singer model for acceleration. The derivation is in a manner completely analogous to that of the Singer model. This model has a higher dimension than the Singer acceleration model and the performance improvement shown in [60, 61] is not convincing because the test scenario does not appear realistic.

*Non-zero-Mean Jerk Model*: Note first that although the Singer model is usually regarded as a model for the target acceleration, it can also be interpreted as a zero-mean jerk model in which the target jerk $\dot{\tilde{a}}(t)$ is defined by $\dot{\tilde{a}}(t) = -\alpha\tilde{a}(t) + w(t)$ with $\tilde{a}(0) = 0$, which is a zero-mean process. A non-zero-mean jerk model was proposed in [62] by introducing an additional term in the Singer-model equation

$$\dot{a}(t) = -\alpha a(t) + w(t) + \bar{\dot{a}} \tag{55}$$

where $\bar{\dot{a}}$ is a non-zero expected jerk. Note that this is not equivalent to assuming $\dot{a}(t) = \dot{\tilde{a}}(t) + \bar{\dot{a}}$ directly, where $\dot{\tilde{a}}(t)$ is the zero-mean Singer jerk process. It was proposed in [62] to determine the mean jerk $\bar{\dot{a}}$ adaptively from the most recent estimates of the target velocity and acceleration under an assumed coordinated-turn maneuver, as discussed in detail in Section VIC. The noise level of $w$ was also proposed to be determined adaptively from the most recent estimate of target orientation and expected maneuver level. This model and the "current" model are the same in spirit—they aim at improving the Singer model by adding a non-zero-mean term to equations that describe the acceleration. As is clear from a comparison of (32) and (55), they would be equivalent if $\bar{\dot{a}} = \alpha\bar{a}(t)$, where $\bar{a}(t)$ is the assumed piecewise constant mean of the acceleration. Consequently, their difference lies mainly in how mean jerk $\bar{\dot{a}}$ and mean acceleration $\bar{a}$ are obtained.

The above model was refined in [63] for the coordinated-turn maneuvers by removing the acceleration mean to yield

$$\dot{a}(t) = -\alpha[a(t) - \bar{a}(t)] + w(t) + \bar{\dot{a}}. \tag{56}$$

On the other hand, since mean jerk is equal to the derivative of the expected acceleration, the noiseless part of this equation also follows from taking derivative on both sides of $a(t) = \tilde{a}(t) + \bar{a}(t)$, where $\tilde{a}(t)$ is the Singer acceleration. Thus, this model and the "current" model differ only in that the expected acceleration $\bar{a}(t)$ here is not assumed constant over each sampling interval.

# V. 2D HORIZONTAL MOTION MODELS

Most 2D and 3D target maneuver models are naturally turn motion models. These models are usually established relying on target kinematics, in contrast to those of the previous section that are based on random processes. This is understandable since random processes are more natural for modeling time correlation than describing spatial trajectories where kinematics is a more appropriate tool.

2D *horizontal* motion models are described in this section generally in an order from the simplest to the

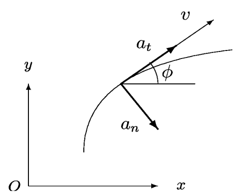

Fig. 6. Geometry of 2D target motion.

most sophisticated. 3D motion models, including those for (nonhorizontal) planar motion in the 3D space, are described in the next section.

Coordinate-coupled target models are highly dependent on the choice of the state components. The choice of the state components (and implicitly the respective kinematic model) is not a trivial problem [64], where target dynamics, accuracy of approximations, sensor coordinate system, among others, must be taken into account.

Various (noiseless part of) kinematic models proposed for tracking of a target moving in the horizontal plane can be comprised from the following standard curvilinear-motion model from kinematics (see Fig. 6):

$$\dot{\mathsf{x}}(t) = v(t)\cos\phi(t) \tag{57}$$

$$\dot{\mathsf{y}}(t) = v(t)\sin\phi(t) \tag{58}$$

$$\dot{v}(t) = a_t(t) \tag{59}$$

$$\dot{\phi}(t) = a_n(t)/v(t) \tag{60}$$

where $(\mathsf{x},\mathsf{y}), v, \phi$ are the target position in Cartesian coordinates, ground speed (airspeed plus wind speed), and (velocity) heading angle, respectively, and $a_t$ and $a_n$ are the target tangential (along-track) and normal (cross-track) accelerations in the horizontal plane, respectively. This model is fairly general—it accounts for along- and cross-track accelerations, and reduces to the following special cases:

1) $a_n = 0$, $a_t = 0$—rectilinear, CV motion,
2) $a_n = 0$, $a_t \neq 0$—rectilinear, accelerated motion (CA motion if $a_t$ = constant),
3) $a_n \neq 0$, $a_t = 0$—circular, constant-speed motion (CT motion if $a_n$ = constant).

The last case above with a constant $a_n$, which has a constant speed and a constant turn rate, is referred to as a constant turn (CT) in this survey but is often referred to as a coordinated turn (CT) in target tracking, due to an abuse of terminology in our opinion. The motion referred to as a coordinated turn in aviation is in fact not so simple and limited, which refers to certain constraints in terms of flight *dynamics*, rather than this purely *kinematic* model. (This is addressed in more detail in Section VIC, including a more reasonable definition in target tracking.) Such motion is preferably specified in terms of the turn rate $\omega = \dot{\phi}$.

The following variant of (60) was used in [65, 66]: $\dot{\phi} = a_n^*/v$, where $a_n = a_n^*\sin(-\psi)$ is the projection of the actual normal acceleration $a_n^*$ on the horizontal plane and $\psi$ is the roll (i.e., bank) angle. This variant makes explicit the relationship between the normal acceleration in the horizontal plane and the actual normal acceleration in a bank-to-turn horizontal motion.

## A. CT Models with Known Turn Rate

These models presume that the target moves with (nearly) constant speed $v$ and (nearly) constant angular (turn) rate $\omega$. Assuming $\omega$ is *known* leads to (only four-dimensional) state vector, e.g., $x = [\mathsf{x},\dot{\mathsf{x}},\mathsf{y},\dot{\mathsf{y}}]'$, in the Cartesian coordinates. It follows immediately from (57)–(60) that such a circular motion is described by (5) with $f(x,u,t) = [\dot{\mathsf{x}}, -\omega\dot{\mathsf{y}}, \dot{\mathsf{y}}, \omega\dot{\mathsf{x}}]'$; that is,

$$\dot{x}(t) = \begin{bmatrix} \dot{\mathsf{x}}(t) \\ -\omega\dot{\mathsf{y}}(t) \\ \dot{\mathsf{y}}(t) \\ \omega\dot{\mathsf{x}}(t) \end{bmatrix} + Bw(t) = A(\omega)x(t) + Bw(t)$$
$$A(\omega) = \begin{bmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & -\omega \\ 0 & 0 & 0 & 1 \\ 0 & \omega & 0 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 & 0 \\ 1 & 0 \\ 0 & 0 \\ 0 & 1 \end{bmatrix} \tag{61}$$

where white noise $w = [w_x, w_y]'$ has the power spectral density $\mathrm{diag}[S_w, S_w]$. This CT model is linear since $\omega$ is known. Its discrete-time equivalent is found to be

$$x_{k+1} = F_{ct}(\omega)x_k + w_k = \begin{bmatrix} 1 & \dfrac{\sin\omega T}{\omega} & 0 & -\dfrac{1-\cos\omega T}{\omega} \\ 0 & \cos\omega T & 0 & -\sin\omega T \\ 0 & \dfrac{1-\cos\omega T}{\omega} & 1 & \dfrac{\sin\omega T}{\omega} \\ 0 & \sin\omega T & 0 & \cos\omega T \end{bmatrix} x_k + w_k \tag{62}$$

where

$$Q = \mathrm{cov}(w_k) = S_w \begin{bmatrix} \dfrac{2(\omega T - \sin\omega T)}{\omega^3} & \dfrac{1-\cos\omega T}{\omega^2} & 0 & \dfrac{\omega T - \sin\omega T}{\omega^2} \\ \dfrac{1-\cos\omega T}{\omega^2} & T & -\dfrac{\omega T - \sin\omega T}{\omega^2} & 0 \\ 0 & -\dfrac{\omega T - \sin\omega T}{\omega^2} & \dfrac{2(\omega T - \sin\omega T)}{\omega^3} & \dfrac{1-\cos\omega T}{\omega^2} \\ \dfrac{\omega T - \sin\omega T}{\omega^2} & 0 & \dfrac{1-\cos\omega T}{\omega^2} & T \end{bmatrix}. \tag{63}$$

Its direct discrete-time counterpart (see [67, 1, 2]) is better known, given by (62), where $w_k$ is replaced by $\mathrm{diag}[G_2, G_2] w_k$ with $G_2$ defined in (14) and a directly defined $\mathrm{cov}(w_k)$. The (zero-mean, Gaussian, white) noise $w$ in the above is used to model the perturbation of the trajectory from the ideal CT motion.

An approximation[12] of the noiseless part of (62) is [68, 69]:

$$F_{ct}(\omega) x_k \approx \begin{bmatrix} 1 & T & 0 & -\omega T^2/2 \\ 0 & 1-(\omega T)^2/2 & 0 & -\omega T \\ 0 & \omega T^2/2 & 1 & T \\ 0 & \omega T & 0 & 1-(\omega T)^2/2 \end{bmatrix} x_k = \begin{bmatrix} \mathsf{x} + [\dot{\mathsf{x}} - (1/2)\dot{\mathsf{y}}\omega T]T \\ \dot{\mathsf{x}} - \dot{\mathsf{y}}\omega T - (1/2)\dot{\mathsf{x}}(\omega T)^2 \\ \mathsf{y} + [\dot{\mathsf{y}} + (1/2)\dot{\mathsf{x}}\omega T]T \\ \dot{\mathsf{y}} + \dot{\mathsf{x}}\omega T - (1/2)\dot{\mathsf{y}}(\omega T)^2 \end{bmatrix} \tag{64}$$

which is a second-order polynomial in $\omega$. It provides a simple but less accurate alternative to the exact CT model. It is of certain value when a nonlinear tracker (e.g., extended Kalman filter (EKF)) is designed with a state vector that includes the (unknown) turn rate. However, it is valid only for $\omega T \approx 0$, which may be violated in many cases with large sampling interval $T$, such as in an air traffic control (ATC) system.

In the rare cases where the constant turn rate is (approximately) known a priori, the above CT model gives good tracking performance. The necessity of an exact knowledge about the value of the turn rate makes the direct use of this model unrealistic for most practical applications. A natural idea is to replace the above $\omega$ by its estimate, based on, e.g., the latest velocity estimates, as used in [70, 67, 4]. However, this may inject unacceptably large errors into the system. Additional efforts are obviously requisite to model the motion with an unknown turn rate within this framework.

*Multiple Known Turn-Rate Models*: Another natural solution is based on the use of multiple models with different, fixed turn rates. This approach alleviates the effect of the uncertainty in the turn rate and takes advantage of the simple and linear form of the dynamic model (73) given the turn rate. In this approach, the sequence of turn rates $\langle \omega_k \rangle$ is modeled as a Markov (or semi-Markov) chain taking values in the set $\{\omega_1, \omega_2, \ldots, \omega_n\}$, governed by the transition probabilities $P\{\omega_k = \omega_i \mid \omega_{k-1} = \omega_j\}$, $i, j = 1, \ldots, n$, as well as initial probabilities. This approach has been well established (see, e.g., [67, 4], and [68, 69]), mostly for ATC tracking applications. Therefore, a main application of this CT model with known turn rate is serving as one or more (with different $\omega$ values) models in a multiple-model architecture.

[12] By expanding $\sin\theta$ and $\cos\theta$ up to the second-order terms: $\sin\theta \approx \theta$ and $\cos\theta \approx 1 - \theta^2/2$ for $\theta \approx 0$.

## B. CT Models with Unknown Turn Rate

These models differ from the above CT models only in that the turn rate is included as a state component, to be estimated. As such, they are described by (61) in continuous-time or (62) in discrete-time plus an additional equation for $\omega$. The two most popular models for $\omega$ are the *Wiener process model*

$$\dot{\omega}(t) = w_\omega(t), \quad \text{in continuous time} \tag{65}$$

$$\omega_{k+1} = \omega_k + w_{\omega,k}, \quad \text{in discrete time} \tag{66}$$

and the *first-order Markov process model*

$$\dot{\omega}(t) = -\frac{1}{\tau_\omega}\omega(t) + w_\omega(t), \quad \text{in continuous time} \tag{67}$$

$$\omega_{k+1} = e^{-T/\tau_\omega}\omega_k + w_{\omega,k}, \quad \text{in discrete time} \tag{68}$$

where $\tau_\omega$ is the correlation time constant for the turn rate, and $w$ is zero-mean white noise of a suitable level, which can be determined exactly the same way as for the corresponding models for acceleration described in the previous section. Some other models described in the previous section can also be used for the turn rate. For example, the renewal model of [57] is one of them.

Discretization of a continuous-time model here has a unique issue: which discrete-time $\omega$ should be used in $F_{ct}(\omega)$ of (62)? The most popular way is to use $\omega_k$ here. Alternatively, it was proposed in [71] to use $\omega_{k+1}$ in $F_{ct}(\omega)$ instead. This implicit method is more stable. The difference of the two schemes is somewhat similar in spirit to that of approximating a derivative by the forward difference and backward difference in the finite difference method. With this rough analogy in mind, it appears that replacing $\omega$ in (62) by $\bar{\omega} = \frac{1}{2}(\omega_k + \omega_{k+1})$ would further improve accuracy since the center point is usually a better approximation than either of the end points, more or less like the central difference is more accurate than the forward difference and backward difference at the cost of more computation (and possibly numerical instability). For the current problem, however, no extra computation is required by this new scheme if the procedure of solving the linearized equation as proposed in [71] is used, although possible numerical problems should be checked. These three schemes lead to the following three different linearized models by the first-order Taylor series expansions at $[x, \omega]' = [\hat{x}_{k|k}, \hat{\omega}_{k|k}]'$:

$$\begin{aligned} F_{ct}(\omega_k)x_k &= F_{ct}(\hat{\omega}_{k|k})x_k + F_\omega(\hat{\omega}_{k|k})\hat{x}_{k|k}(\omega_k - \hat{\omega}_{k|k}) \\ F_{ct}(\omega_{k+1})x_k &= F_{ct}(\hat{\omega}_{k|k})x_k + F_\omega(\hat{\omega}_{k|k}) \\ &\quad \times \hat{x}_{k|k}(\omega_{k+1} - \hat{\omega}_{k|k}) \\ F_{ct}(\bar{\omega})x_k &= F_{ct}(\hat{\omega}_{k|k})x_k + F_\omega(\hat{\omega}_{k|k})\hat{x}_{k|k}(\bar{\omega} - \hat{\omega}_{k|k}) \end{aligned} \tag{69}$$

where $F_\omega(\hat{\omega}_{k|k}) = (\partial/\partial\omega)F_{ct}(\omega)\,|_{\omega=\hat{\omega}_{k|k}}$. If (66) is used, however, these models have an identical state prediction because

$$E[x_{k+1} \mid z^k] = E[F_{ct}(\omega)x_k \mid z^k] = F_{ct}(\hat{\omega}_{k|k})\hat{x}_{k|k} \quad (70)$$

where $z^k$ stands for all measurements through time $t_k$. This follows from $E[\bar{\omega} \mid z^k] = E[\omega_{k+1} \mid z^k] = E[\omega_k \mid z^k] = \hat{\omega}_{k|k}$, a consequence of (66). Nevertheless, the corresponding covariances differ because different models are used. That is why these models result in very similar tracking performance, as reported in [72] without explanation, for two IMM-EKF (interacting multiple model) algorithms using the first two schemes above with real ATC data. The state prediction would also be different if model (68) is used.

We point out that it is theoretically more appealing to obtain a discrete-time model for state vector $x = [\mathsf{x},\dot{\mathsf{x}},\mathsf{y},\dot{\mathsf{y}},\omega]'$ by discretizing the following continuous-time equations *jointly*:

$$[\dot{\mathsf{x}},\ddot{\mathsf{x}},\dot{\mathsf{y}},\ddot{\mathsf{y}}]'(t) = (\dot{\mathsf{x}},-\omega\dot{\mathsf{y}},\dot{\mathsf{y}},\omega\dot{\mathsf{x}})'(t) + w(t) \quad (71)$$

$$\dot{\omega}(t) = -\alpha\omega(t) + w_\omega(t) \quad (72)$$

where $\alpha = 0$ or $1/\tau_\omega$. It is reasonable to expect an improvement in performance with this alternative at the price of an increased complexity.

A continuous-time model is more accurate than its discrete-time versions, but the latter are needed for most applications. In the case where the continuous-time model is nonlinear, such as the CT models with unknown turn rate, there are in general at least two approaches to acquiring its discrete-time linear approximate models. Such models are needed in the application of a linear (e.g., Kalman) filter based nonlinear filtering technique. The first is to linearize the nonlinear equation for the state first and then discretize the linearized differential equation (by finding the solution via integration). This approach is more commonly used because it is easy. An alternative is to discretize the nonlinear state-space equation first and then linearize the discrete-time model. These two approaches are referred to as *discretized linearization* and *linearized discretization*, respectively, in [73, 64]. The second approach appears more accurate in general than the first since linearization usually lose more accuracy than discretization and thus should be done later. However, the second approach may not be tractable in general because discretization requires solving the nonlinear differential equation, which is often a great challenge. Fortunately, for the CT motion with an unknown turn rate, the corresponding differential equations for the state are simple and the solution can be readily obtained [73, 64]. The equations of the nearly CT model, obtained by the first approach, and some performance comparison results are also given in [73, 64].

As stated before, the choice of the state vector is not trivial for the turn models. Essentially two classes have been proposed. They differ in the representation of the velocity vector: in the Cartesian and polar coordinates, respectively.

1) *CT Models with Cartesian Velocity*: In this model, the state vector is chosen to be $x = [\mathsf{x},\dot{\mathsf{x}},\mathsf{y},\dot{\mathsf{y}},\omega]'$; that is, the velocity vector $(\dot{\mathsf{x}},\dot{\mathsf{y}})'$ is represented in the Cartesian coordinates. Consequently, a direct discrete-time version of the model is given by (see [67, 2]):

$$x_{k+1} = \begin{bmatrix} F_{ct}(\omega^*) & 0 \\ 0 & \beta \end{bmatrix} x_k + \mathrm{diag}[G_2, G_2, 1]w_k \quad (73)$$

where $\beta = e^{-\alpha T}$, $\omega^* = \omega_k$, $\omega_{k+1}$, $\bar{\omega}$, or something similar, $G_2$ was given in (14), $w_k = [w_x, w_y, w_\omega]'_k$ is zero-mean white noise with suitable statistics—they are noise terms for acceleration in the $\mathsf{x}$ and $\mathsf{y}$ directions and for turn rate.

This model is known to be used successfully as one of models in numerous multiple-model configurations (see, e.g., [74, 67, 2, 75, 4]).

2) *CT Models with Polar Velocity*: Obviously, the velocity vector can also be represented in the polar coordinates as $[v,\phi]'$ in terms of speed $v = \sqrt{\dot{\mathsf{x}}^2 + \dot{\mathsf{y}}^2}$ and (velocity) heading angle $\phi = \tan^{-1}(\dot{\mathsf{y}}/\dot{\mathsf{x}})$. The state vector is thus $x = [\mathsf{x},\mathsf{y},v,\phi,\omega]'$. The corresponding differential equation is given by (5) with $f(x,u,t) = [v\cos\phi,\ v\sin\phi,\ 0,\ \omega,\ 0]'$; that is

$$\dot{x}(t) = [v\cos\phi, v\sin\phi, 0, \omega, 0]' + w(t) \quad (74)$$

which follows immediately from (57)–(60) by setting $\omega$ and $v$ constant. By linearization first and then discretization, its discrete-time linearized model is given in [76] (see also [1]):

$$x_{k+1} = \begin{bmatrix} \mathsf{x} + (2/\omega)v\sin(\omega T/2)\cos(\phi + \omega T/2) \\ \mathsf{y} + (2/\omega)v,\sin(\omega T/2)\sin(\phi + \omega T/2) \\ v \\ \phi + \omega T \\ \omega \end{bmatrix}_k + w_k \quad (75)$$

where $w_k$ is white noise with covariance

$$Q = \mathrm{diag}\left[\begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, T^2\sigma_{\dot{v}}^2, \begin{bmatrix} T^3\sigma_{\dot{\omega}}^2/3 & T^2\sigma_{\dot{\omega}}^2/2 \\ T^2\sigma_{\dot{\omega}}^2/2 & T^2\sigma_{\dot{\omega}}^2 \end{bmatrix}\right]. \quad (76)$$

This model was successfully used as a building block of a multiple-model algorithm for aircraft tracking application in an air defense system [77, 78].

The above model uses (65) for the turn rate per se. Instead, it may be better to use (67). The corresponding discrete-time model need be modified accordingly.

The discretized linearization (i.e., discretization after linearization) alternative of this model can be found in [73, 64], along with a comparison of its performance with that of the linearized discretization based on a theoretical error analysis[13] and Monte Carlo simulations. It was concluded therein that whenever possible linearized discretization is preferable to discretized linearization. For the two CT models with polar velocity, the former slightly outperforms the latter. It was also claimed therein that the CT model with polar velocity outperforms the CT model with Cartesian velocity.

The use of the following variant of the above CT model with polar velocity was reported in [79]. Let the state vector be $x = [\mathsf{x},\mathsf{y},v,\phi,a_n]'$; that is, replace the turn rate $\omega$ in the above model with the normal (transversal) acceleration $a_n$. Then, it follows immediately from (57)–(60) that the continuous-time state-space model for the CT motion is given by (5) with $f(x,u,t) = [v\cos\phi,\ v\sin\phi,\ 0,\ a_n/v,\ 0]'$. A discrete-time linear approximation of this model was obtained based on a refined Euler-Cauchy scheme as [79]

$$x_{k+1} = x_k + Tf(x_k + \tfrac{1}{2}Tf(x_k)) + w_k \tag{77}$$

followed by the standard EKF linearization, where $w_k$ is white noise, obtained by a backward difference of (the derivative of) the continuous-time white noise. Note that $x_{k+1/2} \approx x_k + \frac{1}{2}Tf(x_k)$. A set of such unknown acceleration models was included in a tracker for ATC in [79] to account for possible horizontal motions.

### C. Circular Motion Models

For a circular motion of a target, if its center were known, the simplest model would be to represent the circle in the polar coordinates and place the origin at the circle center. In this coordinate system, the target dynamic model is linear for $x = [\rho,\theta,\dot{\theta}]'$

$$x_{k+1} = \mathrm{diag}[1,F_2]x_k + \mathrm{diag}[1,G_2/T]w_k \tag{78}$$

where $F_2$ and $G_2$ were given by (14) and $w_k$ is white noise. The corresponding measurement equation is pseudo-linear because the noise covariance is actually state dependent, described in detail in a subsequent part (see [15] for a preliminary version). As a result, the Kalman filter is not optimal but can be nonetheless implemented straightforwardly. This *maneuver-centered* CT model was introduced in [80]. While the idea underlying this model is intuitively appealing, the inherent nonlinearity of the problem is not avoided. It obviously relies on an accurate determination of the center of the turn in terms of the sensor coordinate system, which is inherently a nonlinear problem. The following simple geometrically oriented procedure of estimating the center was proposed in [80]. Assume that each target position measurements are points on the circle; replace the chord between any two consecutive measurement points with the straight line segment connecting them; the center can then be determined from the (average) intersection of the perpendicular bisectors of two or more such straight line segments. An essentially the same procedure was used in [81] for estimating the center. Note that using the center estimates injects additional nonlinearities into the system, which are not accounted for in the above linear model.

### D. Curvilinear Motion Model

This model, proposed in [82], is more general than those considered so far in this section. It accounts for possibly non-zero normal (cross-track) and tangential (along-track) target maneuver accelerations simultaneously. For the Cartesian state vector $x = [\mathsf{x},\dot{\mathsf{x}},\mathsf{y},\dot{\mathsf{y}}]'$, it follows from the standard equations of curvilinear motion (57)–(60)[14] that this model in continuous time is given by

$$\dot{x}(t) = A_{cv}x(t) + B(x(t))a(t) + w(t) \tag{79}$$

where $a = [a_t,a_n]'$ is the acceleration for the maneuver, $A_{cv}$ was given by (11), and

$$B(x(t)) = \begin{bmatrix} 0 & 0 \\ \dfrac{\dot{\mathsf{x}}(t)}{\sqrt{\dot{\mathsf{x}}(t)^2+\dot{\mathsf{y}}(t)^2}} & -\dfrac{\dot{\mathsf{y}}(t)}{\sqrt{\dot{\mathsf{x}}(t)^2+\dot{\mathsf{y}}(t)^2}} \\ 0 & 0 \\ \dfrac{\dot{\mathsf{y}}(t)}{\sqrt{\dot{\mathsf{x}}(t)^2+\dot{\mathsf{y}}(t)^2}} & \dfrac{\dot{\mathsf{x}}(t)}{\sqrt{\dot{\mathsf{x}}(t)^2+\dot{\mathsf{y}}(t)^2}} \end{bmatrix}. \tag{80}$$

Pretending the solution formula for the linear system works for this actually nonlinear case, its discretized version (by the standard discretization) in the form

$$x_{k+1} = F_{cv}x_k + G_k(x)a_k + w_k \tag{81}$$

is highly nonlinear because of its dependence on the target state via matrix $B$:

$$G_k(x) = \int_0^T e^{A(T-\tau)}B(x(kT+\tau))d\tau \tag{82}$$

and the integration involved in $G_k(x)$ is hard to evaluate exactly. It was shown in [82] that $G_k(x)$ can be found approximately to be, with $\varphi_{k+1} \stackrel{\Delta}{=} \phi_k + \omega_k T$,

[13]Based on estimation of the Frobenius norm of the neglected terms in the approximations.

[14]The heading angle $\phi$ defined in [82] differs from the traditional one, which is adopted here.

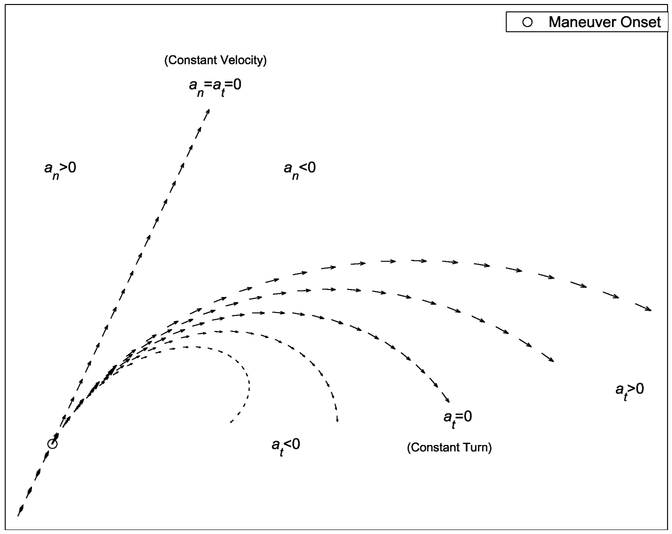


Fig. 7. Curvilinear motion trajectories.

$$G_k(x) \approx G_a(\phi_k,\omega_k) = \left[ G_{a_t}(\phi_k,\omega_k), \begin{bmatrix} \frac{1}{\omega_k^2}\sin\varphi_{k+1} - \frac{1}{\omega_k^2}\sin\phi_k - \frac{1}{\omega_k}T\cos\phi_k \\ \frac{1}{\omega_k}\cos\varphi_{k+1} - \frac{1}{\omega_k}\cos\phi_k \\ -\frac{1}{\omega_k^2}\cos\varphi_{k+1} + \frac{1}{\omega_k^2}\cos\phi_k - \frac{1}{\omega_k}T\sin\phi_k \\ \frac{1}{\omega_k}\sin\varphi_{k+1} - \frac{1}{\omega_k}\sin\phi_k \end{bmatrix} \right] \tag{83}$$

$$G_{a_t}(\phi_k,\omega_k) = \begin{bmatrix} -\frac{1}{\omega_k^2}\cos\varphi_{k+1} + \frac{1}{\omega_k^2}\cos\phi_k - \frac{1}{\omega_k}T\sin\phi_k \\ \frac{1}{\omega_k}\sin\varphi_{k+1} - \frac{1}{\omega_k}\sin\phi_k \\ -\frac{1}{\omega_k^2}\sin\varphi_{k+1} + \frac{1}{\omega_k^2}\sin\phi_k + \frac{1}{\omega_k}T\cos\phi_k \\ -\frac{1}{\omega_k}\cos\varphi_{k+1} + \frac{1}{\omega_k}\cos\phi_k \end{bmatrix} \tag{84}$$

under the following simplifying assumptions: 1) the acceleration $a$ is piecewise constant over each sampling interval $[kT,\ kT+T)$ and 2) the speed change over a sampling interval is much smaller than the speed itself: $a_{t_k}T \ll v_k$. An analytically equivalent form[15] of (81) is, with $a_t$ being the only explicit forcing term,

$$x_{k+1} = F_{ct}(\omega_k)x_k + G_{a_t}(\phi_k,\omega_k)a_{t_k} + w_k. \tag{85}$$

Note that while in (81) the maneuver acceleration term $G_a(\phi_k,\omega_k)a_k$ is "added" to the CV motion, in (85) the effect of the tangential acceleration $a_t$ [i.e., the $G_{a_t}(\phi_k,\omega_k)a_{t_k}$ term] is "added" to the CT (constant-speed constant turn-rate) motion. Equation (85) makes it clear the capability of this model to account for "relatively small" tangential accelerations as well as the normal accelerations, while the popular CT model accounts only for the latter (see Fig. 7). Both forms are applicable to tracking targets performing maneuvers with concurrent non-zero along- and cross-track accelerations, but attention should be paid to the approximating assumptions of the models.

This model, combined with a suitable model for turn rate, is one of the most sophisticated target maneuver models for 2D horizontal motions.

[15]It can be obtained by using $\dot{\mathsf{x}}_k = (a_n/\omega_k)\cos\phi_k$ and $\dot{\mathsf{y}}_k = (a_n/\omega_k)\cos\phi_k$.

## E. Ship Models

The maneuver models surveyed in this paper, especially those for 2D horizontal motions and all 3D models of the next section, are particularly suitable for aircraft. These models were developed for target tracking purposes. For other (e.g., navigation) purposes, numerous, more precise dynamic models can be found in the literature on vehicle dynamics (see, e.g., [83–86]), which is beyond the scope of this survey. These models, however, require better knowledge about the vehicle than what is available to a tracker. A few of these models have been adapted to tracking applications, resulting in vehicle dynamics based models. An example is those for aircraft based on flight dynamics described in Section VID. Note, however, that they are not really point-target models.

Likewise, a number of precise ship motion models based on ship dynamics are available in the literature, which depend on the particular ship's form and size. There exist also less precise but more generally applicable models. The following model from [87] is one in the latter class

$$\dot{\mathsf{x}} = v\sin(\phi - \beta) \tag{86}$$

$$\dot{\mathsf{y}} = v\cos(\phi - \beta) \tag{87}$$

$$\dot{\phi} = \omega \tag{88}$$

$$\omega = K\Omega \tag{89}$$

$$\dot{\Omega} = -\frac{v_0^2}{2pL^2}\left[\frac{qL}{v_0}\Omega + s_{31}\delta\right] \tag{90}$$

$$\dot{\beta} = -\frac{v_0}{2pL}[q\beta + s_{21}\delta] \tag{91}$$

$$v = Kv_0 \tag{92}$$

$$K = \left(1 + \frac{1.9\Omega^2L^2}{v_0^2}\right)^{-1} \tag{93}$$

where model noise is not included for simplicity. Here $(\mathsf{x},\mathsf{y})$, $\phi$, $\omega$, $\Omega$, $\beta$, $\delta$ are ship position, heading, (heading) turn rate, velocity vector turn rate, drift angle, and control ruder angle deviation, respectively;[16] $v = v(\Omega)$ and $v_0 = v(0)$ are ship speeds at turn rate $\Omega$ and $\Omega = 0$ (i.e., at the onset of the turn), respectively; the hydrodynamic constants $p, q, s_{21}, s_{31}$ depend on ship geometry and size, in particular, ship length $L$. The main feature of this model of a ship, which is a sizable object, is revealed by (89) that relates two turn rates and by (92) that relates two speeds. The discretized version of this model with $\beta = 0$ is given by [88, 89, 66]

$$\mathsf{x}_{k+1} = \mathsf{x}_k + Tv_k\sin\phi_k \tag{94}$$

$$\mathsf{y}_{k+1} = \mathsf{y}_k + Tv_k\cos\phi_k \tag{95}$$

$$\phi_{k+1} = \phi_k + Tv_k[\Omega_k + \tfrac{1}{2}(\Omega_k - \Omega_0)Tv_k\tau e^{Tv_k\tau}] \tag{96}$$

$$\Omega_{k+1} = \Omega_k e^{Tv_k\tau} + \Omega_0(1 - e^{Tv_k\tau}) \tag{97}$$

$$v_k = Kv_0 = v_0(1 + 1.9\Omega_k^2L^2)^{-1} \tag{98}$$

where $\tau = (-p/2 + \sqrt{p^2/4 - q})/L$ and $\Omega_0 = \Omega/v_0$. The time constant $\tau$ was set to zero in [88, 90], resulting in a constant turn rate (i.e., $\Omega_{k+1} = \Omega_k$), to eliminate the dependence of the model on the ship-specific hydrodynamic constants. However, the ship length $L$ was actually treated as known therein. The unknown $\Omega_k$ was assumed to take on one of the three possible values $\{0, \Omega_c, -\Omega_c\}$ with a preset constant $\Omega_c$, representing rectilinear, left-turn, and right-turn motions, respectively, and the tracker presented therein was based on a multiple-model algorithm using these three models for $\Omega_k$.

The above nonlinear model has been proposed for ship tracking [88, 89, 66]. Other ship dynamic models are available (see, e.g., [91, 33, 92]), some of which appear simpler and more popular.

# VI. 3D MOTION MODELS

Many of the 2D horizontal models reviewed above have been considered for application to 3D tracking of civilian aircraft in ATC systems. Such targets maneuver mostly in a horizontal plane with nearly constant speed and turn rate and have little or limited vertical maneuver, usually performed not at the time of a horizontal turn. Thus, the altitude changes are most often modeled independently by a (nearly) CV model or a random walk model along $\mathsf{z}$ direction, leading to an acceptable accuracy in practice. However, when the task is to track agile military aircraft, capable of performing "high-$g$" turns in the 3D space (e.g., for tracking in air defense systems) rather than just horizontally, decoupled models may be inadequate. Many efforts have been devoted to solving this problem, and more accurate models have been developed, which are surveyed next.

## A. Basic Kinematic Relations

Let $\mathbf{p} = \overrightarrow{OP}$, $\mathbf{v} = \dot{\mathbf{p}}$, $\mathbf{a} = \dot{\mathbf{v}} = \ddot{\mathbf{p}}$ be target position, velocity, and acceleration, respectively, in the inertial (Cartesian) frame $\mathcal{I} = O_{xyz}$, where $P$ is the target center. Denote $\mathcal{B} = P_{\xi\eta\zeta}$ as the target body frame.

The angular velocity vector of the target is defined in the body frame as $\mathbf{\Omega}^{\mathcal{B}} = (p,q,r)$ and in an arbitrary frame (e.g., the inertial frame $\mathcal{I}$) as [93, 86]

$$\mathbf{\Omega} = p\boldsymbol{\xi} + q\boldsymbol{\eta} + r\boldsymbol{\zeta} \qquad \text{with} \quad p = \dot{\boldsymbol{\eta}}\cdot\boldsymbol{\zeta}, \quad q = \dot{\boldsymbol{\zeta}}\cdot\boldsymbol{\xi}, \quad r = \dot{\boldsymbol{\xi}}\cdot\boldsymbol{\eta} \tag{99}$$

[16]Heading is the angle of the longitudinal axis and velocity heading is the angle of the velocity vector. We use either term if they coincide or one does not exist (strictly speaking, a point target without a shape has no heading). Turn rate is usually defined as the heading change rate.

where $\boldsymbol{\xi}$, $\boldsymbol{\eta}$, $\boldsymbol{\zeta}$ are expressed in that arbitrary frame. The fundamental relation of kinematics (FRK) [86, 94] states that for any time-varying vector $\mathbf{u}(t)$ we have

$$\frac{d\mathbf{u}^{\mathcal{I}}}{dt} = \frac{d\mathbf{u}^{\mathcal{B}}}{dt} + \boldsymbol{\Omega}_{\mathcal{B}\mathcal{I}} \times \mathbf{u} \tag{100}$$

where $\mathbf{u}^{\mathcal{I}}$ and $\mathbf{u}^{\mathcal{B}}$ are $\mathbf{u}$ expressed in the inertial frame $\mathcal{I}$ and the body frame $\mathcal{B}$, respectively, and $\boldsymbol{\Omega}_{\mathcal{B}\mathcal{I}}$ is the angular velocity vector of $\mathcal{B}$ with respect to $\mathcal{I}$. This relation holds true in all frames. For example,

$$\left(\frac{d\mathbf{u}^{\mathcal{I}}}{dt}\right)^{\mathcal{I}} = \left(\frac{d\mathbf{u}^{\mathcal{B}}}{dt}\right)^{\mathcal{I}} + \boldsymbol{\Omega}_{\mathcal{B}\mathcal{I}}^{\mathcal{I}} \times \mathbf{u}^{\mathcal{I}} \tag{101}$$

$$\left(\frac{d\mathbf{u}^{\mathcal{I}}}{dt}\right)^{\mathcal{B}} = \left(\frac{d\mathbf{u}^{\mathcal{B}}}{dt}\right)^{\mathcal{B}} + \boldsymbol{\Omega}_{\mathcal{B}\mathcal{I}}^{\mathcal{B}} \times \mathbf{u}^{\mathcal{B}} \tag{102}$$

where superscripts $\mathcal{I}$ and $\mathcal{B}$ denote quantities expressed in $\mathcal{I}$ and $\mathcal{B}$ frames, respectively. In the sequel, for simplicity we drop superscript $\mathcal{I}$ and write $\dot{\mathbf{u}}$ and $\ddot{\mathbf{u}}$ for $d\mathbf{u}^{\mathcal{I}}/dt$ and $d^2\mathbf{u}^{\mathcal{I}}/dt^2$, respectively. However, we maintain the less compact notations $d\mathbf{u}^{\mathcal{B}}/dt$ and $d^2\mathbf{u}^{\mathcal{B}}/dt^2$, rather than the misleading $\dot{\mathbf{u}}^{\mathcal{B}}$ and $\ddot{\mathbf{u}}^{\mathcal{B}}$ because the latter are likely to be incorrectly interpreted as $(d\mathbf{u}/dt)^{\mathcal{B}}$ and $(d^2\mathbf{u}/dt^2)^{\mathcal{B}}$, respectively. Note that since the target body frame $\mathcal{B}$ rotates as the target does, we have $\boldsymbol{\Omega} = \boldsymbol{\Omega}_{\mathcal{B}\mathcal{I}}^{\mathcal{I}}$, where $\boldsymbol{\Omega}$ is the angular velocity vector of the target in the inertial frame $\mathcal{I}$.

In what follows, we consider only the important case *where the $\xi$-axis of the body frame is aligned with the velocity vector*, that is, $\boldsymbol{\xi} = \mathbf{v}/v$, where $v = \|\mathbf{v}\|$ is the target speed. Then by the Poisson's formula [93] $\dot{\boldsymbol{\xi}} = \boldsymbol{\Omega} \times \boldsymbol{\xi}$, we have

$$\frac{\dot{\mathbf{v}}v - \mathbf{v}\dot{v}}{v^2} = \boldsymbol{\Omega} \times \frac{\mathbf{v}}{v}$$

that is,

$$\dot{\mathbf{v}} = \dot{v}\frac{\mathbf{v}}{v} + \boldsymbol{\Omega} \times \mathbf{v} = \frac{\mathbf{v}\cdot\dot{\mathbf{v}}}{v^2}\mathbf{v} + \boldsymbol{\Omega} \times \mathbf{v} \tag{103}$$

meaning that the total acceleration is the vector sum of a linear acceleration $(\mathbf{v}/v)\dot{v}$ and an acceleration $\boldsymbol{\Omega} \times \mathbf{v}$ responsible solely for turning. This relation also follows directly from FRK (100) with $\mathbf{u} = \mathbf{v}$ and $\boldsymbol{\xi} = \mathbf{v}/v$. Further, it follows from (103) by straightforward calculus[17] that

$$\boldsymbol{\Omega} = \frac{\boldsymbol{\Omega}\cdot\mathbf{v}}{v^2}\mathbf{v} + \frac{\mathbf{v}\times\mathbf{a}}{v^2}. \tag{104}$$

It turns out that (103) and (104) are equivalent.

It is clear from (104) that the angular velocity is given by

$$\boldsymbol{\Omega} = \frac{\mathbf{v}\times\mathbf{a}}{v^2} \tag{105}$$

if and only if $\boldsymbol{\Omega}\cdot\mathbf{v} = 0$, that is, $\boldsymbol{\Omega} \perp \mathbf{v}$. It follows from (105) that $\boldsymbol{\Omega} \perp \mathbf{a}$ and that $\mathbf{v}$ and $\mathbf{a}$ are in a plane (known as the maneuver plane) orthogonal to $\boldsymbol{\Omega}$ and thus the motion is planar, but not necessarily horizontal, if $\boldsymbol{\Omega}$ has a constant direction. Note that

[17] $\mathbf{v}\times\mathbf{a} = \mathbf{v}\times\dot{\mathbf{v}} = \mathbf{v}\times(\boldsymbol{\Omega}\times\mathbf{v}) = (\mathbf{v}\cdot\mathbf{v})\boldsymbol{\Omega} - (\boldsymbol{\Omega}\cdot\mathbf{v})\mathbf{v} = v^2\boldsymbol{\Omega} - (\boldsymbol{\Omega}\cdot\mathbf{v})\mathbf{v}$.

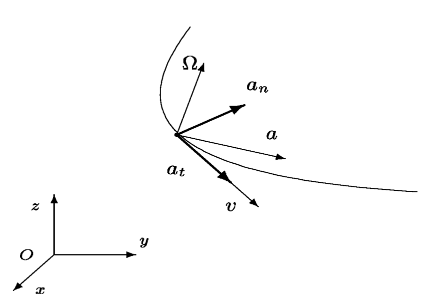

Fig. 8. Geometry of a 3D orthogonal-velocity motion.

this is the case even if $\dot{\boldsymbol{\Omega}} \neq 0$ (i.e., turn rate is not constant) provided $\boldsymbol{\Omega}$ has a constant direction.

Equation (105) plays a central role in most 3D kinematic models. Virtually all models in the literature (e.g., [95, 10, 96, 97, 70, 6, 98, 1, 99]) presume explicitly or implicitly that (105) holds. As shown above, a necessary and sufficient condition for its validity is the orthogonality between the linear and angular velocity vectors. In this sense, models that obey (105) are orthogonal-velocity models, although they are widely referred to as "*coordinated turn*" models, which is actually a misnomer. The geometry of these models is illustrated in Fig. 8.

The FRK (100), (103), and (104) (or (105)) are the key to understand various 3D maneuver models proposed in the literature.

Since the target tracking community is more familiar with the matrix language than the vector analysis and we are more concerned with state-space models, in the sequel we use the following relationship repeatedly. Let $\boldsymbol{\Omega} = [\Omega_x, \Omega_y, \Omega_z]'$ and $\mathbf{v} = [\dot{\mathsf{x}}, \dot{\mathsf{y}}, \dot{\mathsf{z}}]'$. Then

$$\boldsymbol{\Omega} \times \mathbf{v} = M_\Omega \begin{bmatrix} \dot{\mathsf{x}} \\ \dot{\mathsf{y}} \\ \dot{\mathsf{z}} \end{bmatrix}, \qquad M_\Omega = \begin{bmatrix} 0 & -\Omega_z & \Omega_y \\ \Omega_z & 0 & -\Omega_x \\ -\Omega_y & \Omega_x & 0 \end{bmatrix}. \tag{106}$$

## B. Constant-Turn Models

1) *3D Constant-Turn Models*: It is clear from (103) that a constant speed (i.e., $\dot{v} = 0$) motion corresponds to $\mathbf{a}\cdot\mathbf{v} = 0$ (i.e., $\mathbf{a} \perp \mathbf{v}$) and is described equivalently by

$$\mathbf{a} = \boldsymbol{\Omega} \times \mathbf{v}. \tag{107}$$

Since $\boldsymbol{\Omega}$ is unknown in general, one of the simplest possible way is to augment the state vector by its components $\Omega_x, \Omega_y, \Omega_z$ in the inertial frame, to be estimated. That is, let $x = [\mathsf{x}, \mathsf{y}, \mathsf{z}, \dot{\mathsf{x}}, \dot{\mathsf{y}}, \dot{\mathsf{z}}, \Omega_x, \Omega_y, \Omega_z]'$. Then (107) becomes

$$\begin{bmatrix} \ddot{\mathsf{x}} \\ \ddot{\mathsf{y}} \\ \ddot{\mathsf{z}} \end{bmatrix} = \begin{bmatrix} \Omega_x \\ \Omega_y \\ \Omega_z \end{bmatrix} \times \begin{bmatrix} \dot{\mathsf{x}} \\ \dot{\mathsf{y}} \\ \dot{\mathsf{z}} \end{bmatrix} = M_\Omega \begin{bmatrix} \dot{\mathsf{x}} \\ \dot{\mathsf{y}} \\ \dot{\mathsf{z}} \end{bmatrix} = \begin{bmatrix} \Omega_y\dot{\mathsf{z}} - \Omega_z\dot{\mathsf{y}} \\ \Omega_z\dot{\mathsf{x}} - \Omega_x\dot{\mathsf{z}} \\ \Omega_x\dot{\mathsf{y}} - \Omega_y\dot{\mathsf{x}} \end{bmatrix}. \tag{108}$$

If, in addition, $\boldsymbol{\Omega}$ is modeled as nearly constant (i.e., $\dot{\boldsymbol{\Omega}}$ is white noise), then the continuous-time 3D (nearly) CT model—which has (nearly) constant angular velocity and speed in the 3D space—is given in the state space by

$$\dot{x}(t) = \begin{bmatrix} 0 & I_{3\times 3} & 0 \\ 0 & M_\Omega & 0 \\ 0 & 0 & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ I_{3\times 3} \end{bmatrix} w(t). \quad (109)$$

This model is nonlinear since $M_\Omega$ of (106) depends on the state components $\Omega_x, \Omega_y, \Omega_z$.

Note the difference between this model and those based on (105): while $\boldsymbol{\Omega} \perp \mathbf{v}$ under (105), $\boldsymbol{\Omega}$ of (109), albeit nearly constant, is not necessarily orthogonal to $\mathbf{v}$, although $\mathbf{v}$ and $\mathbf{a}$ are orthogonal to each other under (107). $\boldsymbol{\Omega} \perp \mathbf{a}$ is true for both models. To our knowledge, this simple 3D-CT model has not been considered in the tracking literature, despite its generality: it can describe constant-speed motion along non-zero torsion (nonplanar) curves.

This model has a (nearly) constant speed and *angular velocity*. An even more general model can be developed by replacing $\dot{\boldsymbol{\Omega}} = 0$ with $\dot{\omega} = 0$, where $\omega = \|\boldsymbol{\Omega}\|$, resulting in a general constant-turn (constant speed and constant *turn rate*) model.

2) *Planar Constant-Turn Models*: Under an additional assumption $\boldsymbol{\Omega} \perp \mathbf{v}$ [or equivalently, (105)] as well as $\dot{\boldsymbol{\Omega}} = 0$, it follows from differentiation of (107) that

$$\dot{\mathbf{a}} = \boldsymbol{\Omega} \times \mathbf{a} = \boldsymbol{\Omega} \times (\boldsymbol{\Omega} \times \mathbf{v}) = (\boldsymbol{\Omega} \cdot \mathbf{v})\boldsymbol{\Omega} - (\boldsymbol{\Omega} \cdot \boldsymbol{\Omega})\mathbf{v} = -\omega^2 \mathbf{v} \quad (110)$$

where the turn rate $\omega$ is given by

$$\omega \triangleq \|\boldsymbol{\Omega}\| = \frac{\|\mathbf{v} \times \mathbf{a}\|}{v^2} = \frac{\|\mathbf{v}\|\|\mathbf{a}\|}{v^2} = \frac{a}{v}. \quad (111)$$

Therefore, the CT motion can be modeled by a second-order Markov process

$$\dot{\mathbf{a}} = -\omega^2 \mathbf{v} + \mathbf{w} \quad (112)$$

where $\mathbf{w}$ is white noise. As explained in footnote 21 right after (123), this motion is (nearly) planar for small $\mathbf{w}$. This fact, along with $\dot{\boldsymbol{\Omega}} = 0$ and (105), implies that this is a planar constant speed and constant turn rate (or angular velocity) motion. Hence, (112) is a (nearly) planar CT model.

This popular model can be traced back to [95, 100]. A discussion of the vector models (107) and (110), along with an analysis and comparison with similar models, can be found in [99]. These models, along with several implementations, were reviewed in [1].

The state-space form of this model in the Cartesian coordinates is clearly given by, with state $x = [\mathsf{x}, \mathsf{y}, \mathsf{z}, \dot{\mathsf{x}}, \dot{\mathsf{y}}, \dot{\mathsf{z}}, \ddot{\mathsf{x}}, \ddot{\mathsf{y}}, \ddot{\mathsf{z}}]'$,

$$\dot{x}(t) = \begin{bmatrix} 0 & I_{3\times 3} & 0 \\ 0 & 0 & I_{3\times 3} \\ 0 & -\omega^2 I_{3\times 3} & 0 \end{bmatrix} x(t) + \begin{bmatrix} 0 \\ 0 \\ I_{3\times 3} \end{bmatrix} w(t) \quad (113)$$

or with state $x = [\mathsf{x}, \dot{\mathsf{x}}, \ddot{\mathsf{x}}, \mathsf{y}, \dot{\mathsf{y}}, \ddot{\mathsf{y}}, \mathsf{z}, \dot{\mathsf{z}}, \ddot{\mathsf{z}}]'$,

$$\dot{x}(t) = \mathrm{diag}[A(\omega), A(\omega), A(\omega)]x(t) + \mathrm{diag}[B, B, B]w(t) \quad (114)$$

$$A(\omega) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\omega^2 & 0 \end{bmatrix}, \qquad B = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix} \quad (115)$$

where the white noise $w(t)$ has power spectral density matrix $\mathrm{diag}[S_x, S_y, S_z]$. Its discrete-time equivalent model [70] with state $x = [\mathsf{x}, \dot{\mathsf{x}}, \ddot{\mathsf{x}}, \mathsf{y}, \dot{\mathsf{y}}, \ddot{\mathsf{y}}, \mathsf{z}, \dot{\mathsf{z}}, \ddot{\mathsf{z}}]'$ is[18]

$$x_{k+1} = \mathrm{diag}[F(\omega), F(\omega), F(\omega)]x_k + w_k \quad (116)$$

$$F(\omega) = \begin{bmatrix} 1 & \frac{\sin \omega T}{\omega} & \frac{1 - \cos \omega T}{\omega^2} \\ 0 & \cos \omega T & \frac{\sin \omega T}{\omega} \\ 0 & -\omega \sin \omega T & \cos \omega T \end{bmatrix}, \qquad \mathrm{cov}(w_k) = \mathrm{diag}[S_x Q(\omega), S_y Q(\omega), S_z Q(\omega)] \quad (117)$$

$$Q(\omega) = \begin{bmatrix} \frac{6\omega T - 8 \sin \omega T + \sin 2\omega T}{4\omega^5} & \frac{2 \sin^4(\omega T/2)}{\omega^4} & \frac{-2\omega T + 4 \sin \omega T - \sin 2\omega T}{4\omega^3} \\ \frac{2 \sin^4(\omega T/2)}{\omega^4} & \frac{2\omega T - \sin 2\omega T}{4\omega^3} & \frac{\sin^2 \omega T}{2\omega^2} \\ \frac{-2\omega T + 4 \sin \omega T - \sin 2\omega T}{4\omega^3} & \frac{\sin^2 \omega T}{2\omega^2} & \frac{2\omega T + \sin 2\omega T}{4\omega} \end{bmatrix}. \quad (118)$$

[18]The covariance $\mathrm{cov}(w_k)$ given here appears more reasonable than what was suggested in [70, 1].

The motions in the x, y, z directions in this model are coupled only through the common turn rate $\omega$. This model specifies a constant-turn motion in the so-called maneuver plane,[19] defined by the velocity and acceleration vectors.

This model relies on knowledge of $\omega$. If $\omega$ is unknown, a straightforward way is to augment the state vector by $\omega$ and estimate $\omega$ (see Section VB). This, however, increases the state dimension and makes the model highly nonlinear. An alternative is to compute $\omega$ from the best speed and acceleration estimates by (111) so as to take advantage of the linear structure of (116) at a given $\omega$. The use of speed and acceleration *estimates* will, however, inject additional errors into the model and result in accuracy degradation, especially when the orthogonality property $\mathbf{a}\cdot\mathbf{v} = 0$ is severely violated for a constant-speed motion. A possible rescue is to impose $\mathbf{a}\cdot\mathbf{v} = 0$ as a *kinematic constraint*, which, however, turns the dynamic model into a highly nonlinear one if it incorporates the constraint directly. Mainly to avoid this nonlinearity, an alternative approach was adopted in [101, 70, 102, 103] where the constraint was incorporated into a pseudomeasurement model, described in detail in the subsequent part of this survey on measurement models (its conference version is [15]).

## C. Variable-Turn Models

The above models based on the constant-speed condition (107) and the constant turn-rate assumption are restrictive in describing the variety of possible maneuvers. More sophisticated and potentially more accurate models are surveyed next. They do not assume that a target moves with a constant turn and are thus capable of describing more complex maneuvers.

1) *Planar Variable-Turn Model*: By applying the FRK to the target velocity vector $\mathbf{v}$, we have

$$\dot{\mathbf{v}} = \frac{d\mathbf{v}^{\mathcal{B}}}{dt} + \boldsymbol{\Omega}\times\mathbf{v}. \tag{119}$$

Differentiation and applying the FRK again yield

$$\ddot{\mathbf{v}} = \frac{d^2\mathbf{v}^{\mathcal{B}}}{dt^2} + \dot{\boldsymbol{\Omega}}\times\mathbf{v} + 2\boldsymbol{\Omega}\times\dot{\mathbf{v}} - \boldsymbol{\Omega}\times(\boldsymbol{\Omega}\times\mathbf{v}). \tag{120}$$

This equation describes a general motion of a rigid body in space [96]. Under the orthogonal-velocity condition $\boldsymbol{\Omega}\perp\mathbf{v}$, the angular velocity vector $\boldsymbol{\Omega}$ is given by (105). Substituting it in (120) results in modeling the target acceleration as a second-order Markov process with state dependent coefficients[20] in the inertial frame

$$\dot{\mathbf{a}} = -2\alpha\mathbf{a} - (2\alpha^2+\omega^2)\mathbf{v} + \mathbf{w}, \qquad \mathbf{w} = \frac{d^2\mathbf{v}^{\mathcal{B}}}{dt^2} + \dot{\boldsymbol{\Omega}}\times\mathbf{v} \tag{122}$$

where

$$\omega \triangleq \|\boldsymbol{\Omega}\| = \frac{\|\mathbf{v}\times\mathbf{a}\|}{v^2}, \qquad \alpha = -\frac{\mathbf{v}\cdot\mathbf{a}}{v^2}. \tag{123}$$

This model, along with one in a slightly different form in rotating line-of-sight (LOS) coordinates, was proposed in [96]. Both forms were utilized therein for two highly accurate trackers.

The noise term $\mathbf{w}$ in (122) reflects the effect of the forces and moments applied to the target. It was modeled in [96] as zero-mean (Gaussian) white noise with power spectral density matrix $\mathrm{diag}[S_x, S_y, S_z]$ that is to be designed. The damping coefficient $\alpha$ is a normalized target drag (i.e., the ratio of negative tangential acceleration to target speed). Both $\alpha$ and turn rate $\omega$ can be interpreted as unknown model parameters. As stated in [96], it can be shown that (122) defines zero-torsion (planar) trajectories (see Fig. 9) if and only if $\mathbf{w} = 0$, that is, non-zero noise components $\mathbf{w}$ account for out-of-plane motion.[21] As such, (122) is a (nearly) planar variable-turn model if its noise term $\mathbf{w}$ is small.

A possible enhancement of this model would be to model $\mathbf{w}$ in the body frame, which more accurately reflects the nature of the acting forces, and then transform it to the inertial frame.

This planar variable-turn model is more general and perhaps more accurate than the planar CT model of (112). It reduces to the model (112) [with $\omega$ given by (111)] for a CT motion under the constant-speed condition (i.e., $\mathbf{a}\cdot\mathbf{v} = 0$). When the speed is not constant, the damping term due to non-zero $\alpha$ allows the model to automatically adapt itself to target maneuvers. Both $\alpha$ and $\omega$ vanish for an unaccelerated motion.

It is straightforward from (122) to obtain the following continuous-time state-space form of this model in the inertial frame with state $x =$

[19] In the special case where the maneuver plane is horizontal, this model simplifies greatly and it is more commonly used than (the more general) model (57)–(60) for the derivation of some horizontal CT models [1]. In this case (i.e., $a_t = 0$ and $\omega = a_n/v$), however, both models are essentially equivalent.

[20] The formulation given here follows from [96, (5a) and (5b)] in view of the identity

$$\frac{a^2}{v^2} = \left(\frac{va}{v^2}\right)^2(\cos^2\theta + \sin^2\theta) = \left(\frac{\mathbf{v}\cdot\mathbf{a}}{v^2}\right)^2 + \left(\frac{\|\mathbf{v}\times\mathbf{a}\|}{v^2}\right)^2 = \alpha^2 + \omega^2. \tag{121}$$

[21] This implies that as a special case with $\alpha = 0$, (112) with $\mathbf{w} = 0$ describes a planar motion.

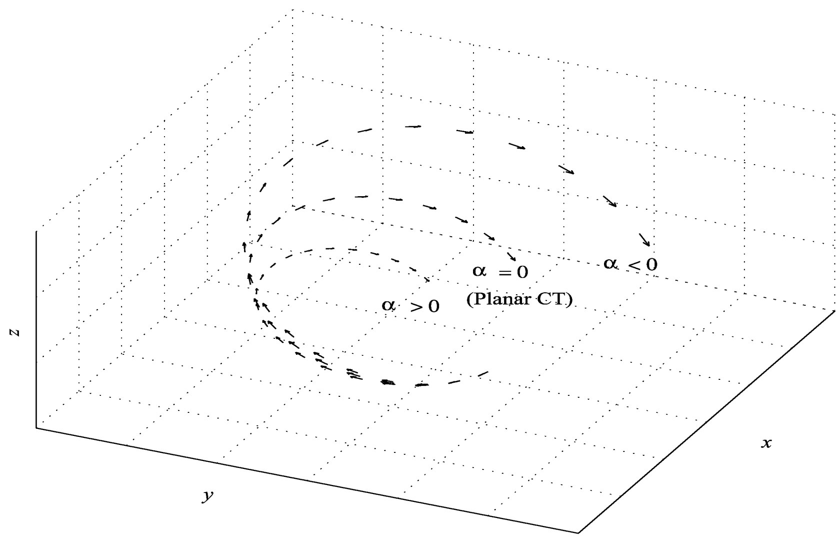


Fig. 9. 3D planar variable-turn trajectories.

$[\mathsf{x},\ \dot{\mathsf{x}},\ \ddot{\mathsf{x}},\mathsf{y},\ \dot{\mathsf{y}},\ \ddot{\mathsf{y}},\mathsf{z},\ \dot{\mathsf{z}},\ \ddot{\mathsf{z}}]'$

$$\dot{x}(t) = \mathrm{diag}[A(\omega_c,\alpha),A(\omega_c,\alpha),A(\omega_c,\alpha)]x(t) + \mathrm{diag}[B_3,B_3,B_3]w(t) \tag{124}$$

$$A(\omega_c,\alpha) = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -(\alpha^2+\omega_c^2) & -2\alpha \end{bmatrix} = \begin{bmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 0 & -\omega_n^2 & -2\zeta\omega_n \end{bmatrix} \tag{125}$$

$$B_3 = \begin{bmatrix} 0 \\ 0 \\ 1 \end{bmatrix}$$

where $\omega_n^2 = \alpha^2 + \omega_c^2$, $\omega_c^2 = \alpha^2 + \omega^2$, and $\zeta = \alpha/\omega_n$, and the white noise $w(t)$ has power spectral density matrix $\mathrm{diag}[S_x,S_y,S_z]$.

It can be seen by comparing (124) with (40) that if the white noise term of this model were replaced by that of (40), the velocity component along each Cartesian coordinate would be implicitly modeled as a random process having an oscillatory exponentially decaying autocorrelation

$$R_{\dot{\mathsf{x}}}(\tau) = R_{\dot{\mathsf{y}}}(\tau) = R_{\dot{\mathsf{z}}}(\tau) = \sigma^2 e^{-\alpha|\tau|}\cos(\omega_c\tau) = \sigma^2 e^{-\zeta\omega_n|\tau|}\cos(\omega_n\sqrt{1-\zeta^2}\tau) \tag{126}$$

Note that both the actual oscillation frequency $\omega_c$ and the undamped natural frequency $\omega_n$ of the velocity autocorrelation functions increase with the damping coefficient $\alpha$ for a given physical turn rate $\omega$ of the target; the frequencies and the turn rate are equal (i.e., $\omega_c = \omega_n = |\omega|$) only if there is no damping; the autocorrelations $R_{\dot{\mathsf{x}}}(\tau) = R_{\dot{\mathsf{y}}}(\tau) = R_{\dot{\mathsf{z}}}(\tau)$ are not periodic when damping is large (i.e., $\zeta \geq 1$). This makes good sense in view of the physical meanings of $\alpha$, $\omega$, $\omega_c$, and $\omega_n$.

Like the planar CT model (116), motions in different coordinates in this model are coupled only through the common $\alpha$ and $\omega_c$. This is another difference from that of (40) with acceleration replaced by velocity, where motions in different coordinates are not coupled (i.e., different values of $\alpha$ and $\omega_c$ may be involved).

We found the discrete-time equivalent model as

$$x_{k+1} = \mathrm{diag}[F(\omega_c,\alpha),F(\omega_c,\alpha),F(\omega_c,\alpha)]x_k + w_k \tag{127}$$

$$Q = \mathrm{cov}(w_k) = \mathrm{diag}[S_xQ(\omega_c,\alpha),S_yQ(\omega_c,\alpha),S_zQ(\omega_c,\alpha)] \tag{128}$$

where $F(\omega_c,\alpha)$ was defined by (46) and $Q(\omega_c,\alpha)$ is given by

$$q_{11} = \frac{A+B+C}{4\alpha\omega^2(\alpha^2+\omega^2)^3}$$

$$A = e^{-2\alpha T}[(\alpha^2-3\omega^2)c + (\omega^2-3\alpha^2)s - (\alpha^2+\omega^2)^2]$$

$$B = 8e^{-\alpha T}\alpha\omega[2\alpha\omega c_0 + (\alpha^2-\omega^2)s_0]$$

$$C = \alpha^2\omega^2(4\alpha T - 11) + \omega^4(1+4\alpha T)$$

$$q_{12} = \frac{e^{-2\alpha T}(\omega c_0 + \alpha s_0 - e^{-\alpha T}\omega)^2}{2\omega^2(\alpha^2+\omega^2)^2}$$

$$q_{13} = \frac{e^{-2\alpha T}(c - s - \alpha^2 + \omega^2) + 4e^{-\alpha T}\alpha\omega s_0 - \omega^2}{4\alpha\omega^2(\alpha^2+\omega^2)}$$

$$q_{22} = \frac{e^{-2\alpha T}(c - s - \alpha^2 - \omega^2) + \omega^2}{4\alpha\omega^2(\alpha^2+\omega^2)}$$

$$q_{23} = \frac{e^{-2\alpha T}s_0^2}{2\omega^2}$$

$$q_{33} = \frac{e^{-2\alpha T}(c + s - \alpha^2 - \omega^2) + \omega^2}{4\alpha\omega^2}$$

$$c = \alpha^2\cos 2\omega T, \qquad s = \alpha\omega\sin 2\omega T$$

$$\omega = \omega_c, \qquad c_0 = \cos\omega T, \qquad s_0 = \sin\omega T.$$

2) *Coordinated-Turn Models*: By the FRK (100) with $\mathbf{u} = \mathbf{a}_g \stackrel{\Delta}{=} \mathbf{a} - \mathbf{g}$, where $\mathbf{g}$ stands for the constant gravitational acceleration in the inertial frame, we have in the body frame

$$(\dot{\mathbf{a}}_g)^{\mathcal{B}} = \left(\frac{d\mathbf{a}_g^{\mathcal{B}}}{dt}\right)^{\mathcal{B}} + (\boldsymbol{\Omega} \times \mathbf{a}_g)^{\mathcal{B}}. \tag{129}$$

Let $T$ be the thrust (minus drag) and $L$ the lift, along $\boldsymbol{\xi}$ and $\boldsymbol{\eta}$ directions, respectively. Then, we have $\mathbf{a}_g^{\mathcal{B}} = [T,\ L,\ 0]'$, $\boldsymbol{\Omega}^{\mathcal{B}} = [p,\ q,\ r]'$, and

$$(\boldsymbol{\Omega} \times \mathbf{a}_g)^{\mathcal{B}} = M_\Omega^{\mathcal{B}} \begin{bmatrix} T \\ L \\ 0 \end{bmatrix}, \qquad M_\Omega^{\mathcal{B}} = \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix}. \tag{130}$$

Since $(\dot{\mathbf{a}}_g)^{\mathcal{B}} = T_{\mathrm{IF}}^{\mathrm{BF}} \dot{\mathbf{a}}_g$, where $T_{\mathrm{IF}}^{\mathrm{BF}}$ is the transform from the inertial frame to the body frame, (129) becomes

$$T_{\mathrm{IF}}^{\mathrm{BF}} \dot{\mathbf{a}}_g = \begin{bmatrix} \dot{T} \\ \dot{L} \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} \begin{bmatrix} T \\ L \\ 0 \end{bmatrix}. \tag{131}$$

An equivalent version of this general model in the body frame for the so-called coordinated turn motion was used in [62] to derive an equation for mean jerk.

A class of motion in the 3D space, known as coordinated turn, is defined (for aircraft) by the following conditions, introduced originally in [62] and refined in [98]: 1) (mean) thrust $T$ (minus drag)—acceleration along the velocity direction—and (mean) lift $L$ (acceleration normal to the aircraft wing plane) are constant; 2) (mean) roll angle $\psi$ (angle around the longitudinal axis) is constant; and 3) (mean) angle of attack and sideslip[22] are zero. These conditions confine the (average) target motion to a plane, known as the *maneuver plane*. They are consistent with the bank-to-turn characteristics of fixed-wing aircraft. Clearly such an (average) motion does not necessarily have a constant turn rate or constant speed.

Under these coordinated turn conditions, we have $\dot{T} = 0$, $\dot{L} = 0$, $p = \dot{\boldsymbol{\eta}} \cdot \boldsymbol{\zeta} = (\boldsymbol{\Omega} \times \boldsymbol{\eta}) \cdot \boldsymbol{\zeta} = 0$ (because $\boldsymbol{\Omega}$, $\boldsymbol{\eta}$, $\boldsymbol{\zeta}$ are in the same plane orthogonal to $\boldsymbol{\xi}$), and thus (131) becomes

$$\dot{\mathbf{a}} = \dot{\mathbf{a}}_g = T_{\mathrm{BF}}^{\mathrm{IF}} \begin{bmatrix} -rL \\ rT \\ -qT \end{bmatrix}, \qquad T_{\mathrm{BF}}^{\mathrm{IF}} = (T_{\mathrm{IF}}^{\mathrm{BF}})^{-1} = (T_{\mathrm{IF}}^{\mathrm{BF}})'. \tag{132}$$

The expectation of this model is the mean jerk equation obtained in [62]. Its state-space representation can be obtained straightforwardly.

[22]The angle of attack is the angle between the heading and the velocity vector projected onto the longitudinal symmetric plane and the sideslip is the angle between the velocity vector and the longitudinal symmetric plane.

This coordinated turn model involves unknown parameters $T$, $L$, $q$, $r$, which can be obtained as follows [62]. By the FRK (100) with $\mathbf{u} = \mathbf{v}$, we have $\dot{\mathbf{v}} = d\mathbf{v}^{\mathcal{B}}/dt + \boldsymbol{\Omega} \times \mathbf{v}$, which in the body frame is

$$\left(\frac{d\mathbf{v}}{dt}\right)^{\mathcal{B}} = \begin{bmatrix} \dot{v} \\ 0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 & -r & q \\ r & 0 & -p \\ -q & p & 0 \end{bmatrix} \begin{bmatrix} v \\ 0 \\ 0 \end{bmatrix} \tag{133}$$

where $v$ is the target speed. On the other hand, $(d\mathbf{v}/dt)^{\mathcal{B}}$ is the total acceleration $\mathbf{a}^{\mathcal{B}}$ in the body frame, that is,

$$\left(\frac{d\mathbf{v}}{dt}\right)^{\mathcal{B}} = \mathbf{a}^{\mathcal{B}} = \begin{bmatrix} T \\ L \\ 0 \end{bmatrix} + \begin{bmatrix} g_\xi \\ g_\eta \\ g_\zeta \end{bmatrix} \tag{134}$$

where $\mathbf{g}^{\mathcal{B}} = [g_\xi, g_\eta, g_\zeta]'$ is the gravitational acceleration in the body frame. It thus follows that

$$\begin{aligned} r &= (L + g_\eta)/v, \qquad q = -g_\zeta/v \\ T &= \dot{v} - g_\xi, \qquad \mathbf{L} = \mathbf{a} - \mathbf{g} - \mathbf{T}. \end{aligned} \tag{135}$$

The target body frame is determined by the (estimated) velocity—to which the thrust $\mathbf{T}$ is parallel—and lift vector $\mathbf{L}$.

Modeling the target accelerations in the body frame is simpler than in the inertial frame. However, the fact that it is in a frame different than the sensor inertial frame induces difficulties. In essence, all ways around these difficulties involve explicitly or implicitly conversion from one frame to the other, which can be done accurately in general only if accurate knowledge of the target position and model parameters is available. In addition, while this model is intuitive appealing, its implementation is rather complicated due to, e.g., its dependence on thrust, lift, and other parameters.

Furthermore, for the purpose of predicting target position in an antiaircraft fire control application (not as part of any filter), the following model in the maneuver plane was employed in [62]. The aircraft motion is described by a standard curvilinear motion model of (57)–(60) *in the maneuver plane* with acceleration $(a_t, a_n)$ determined by the thrust, lift, and gravity as

$$a_t = T + g_t, \qquad a_n = L\cos\psi + g_n \tag{136}$$

where $L\cos\psi$ is the projection of the lift along the normal direction and $\psi$ is the roll angle. Here, $g_t$ and $g_n$ are the gravity components along target velocity and normal directions, respectively, given by

$$g_t = g_\xi \cos\phi + g_\eta \sin\phi, \qquad g_n = -g_\xi \sin\phi + g_\eta \cos\phi \tag{137}$$

where $\phi$ is the target velocity heading in the fixed Cartesian frame in the maneuver plane. Likewise, the target position $(\mathsf{x}, \mathsf{y})$ (and speed $v$) in (57)–(60) are all in the fixed Cartesian frame in the maneuver plane.

Clearly, the approximate curvilinear model of Section VD in the state-space form can be used here in the maneuver plane. However, the small speed-change assumption as well as the piecewise-constant acceleration assumption on which this approximate model is based should be kept in mind. These assumptions are not valid for a large thrust and sampling interval.

3) *Generalized Coordinated-Turn Model*: Applying the FRK (100) to the target acceleration (with the gravity subtracted) $\mathbf{a}_g = \mathbf{a} - \mathbf{g}$ gives

$$\dot{\mathbf{a}} = \frac{d\mathbf{a}_g^{\mathcal{B}}}{dt} + \mathbf{\Omega} \times (\mathbf{a} - \mathbf{g}). \tag{138}$$

Under the orthogonal-velocity condition $\mathbf{\Omega} \perp \mathbf{v}$ [i.e., (105)] and the constant acceleration condition $d\mathbf{a}_g^{\mathcal{B}}/dt = 0$, (138) becomes

$$\dot{\mathbf{a}} = \frac{\mathbf{v} \times \mathbf{a}}{v^2} \times (\mathbf{a} - \mathbf{g}) \tag{139}$$

which is the kinematic equation of [98]. It can be verified that the curvilinear trajectories generated by (139) have zero torsion, that is, they are confined to the maneuver plane.

If the gravity were not extracted, then the orthogonal-velocity condition (105) and the constant acceleration condition $d\mathbf{a}^{\mathcal{B}}/dt = 0$ (not equivalent to $d\mathbf{a}_g^{\mathcal{B}}/dt = 0$) would lead to the following second-order model in terms of the damping coefficient $\alpha$ and turn rate $\omega$

$$\dot{\mathbf{a}} = \frac{\mathbf{v} \times \mathbf{a}}{v^2} \times \mathbf{a} = \frac{\mathbf{v} \cdot \mathbf{a}}{v^2}\mathbf{a} - \frac{a^2}{v^2}\mathbf{v} = -\alpha\mathbf{a} - (\alpha^2 + \omega^2)\mathbf{v} \tag{140}$$

which is similar to (122).

The conditions $\mathbf{\Omega} \perp \mathbf{v}$ and $d\mathbf{a}_g^{\mathcal{B}}/dt = 0$ are closely related with coordinated turns. The constant thrust and lift condition validates assumption $d\mathbf{a}_g^{\mathcal{B}}/dt = 0$; the zero angle of attack and sideslip condition aligns the $\xi$-axis of the real body frame (as defined in flight dynamics) with the assumed velocity-based body frame; and the constant roll angle condition validates assumption $\mathbf{\Omega} \cdot \mathbf{v} = 0$ since the roll angular rate is the projection of the angular velocity to the heading direction: $\dot{\psi} = \mathbf{\Omega} \cdot (\mathbf{v}/v)$. As such, a coordinated turn motion can be readily described by (139) and thus (139) is a generalized model for coordinated turn and quasi coordinated turn motions.

This model was originally proposed in [98], implemented in [104], and later analyzed and validated in [99] by using real data of aircraft trajectories.

It was proposed in [104] to use a (vector-valued) Singer process $\mathbf{u}$ to model the perturbations in acceleration in the body frame. As a result, the state-space representation of this *generalized coordinated turn model* for the state vector $x = [\mathbf{p}', \mathbf{v}', \mathbf{a}', \mathbf{u}']'$ is [104]

$$\dot{x} = \begin{bmatrix} \mathbf{v} \\ \mathbf{a} \\ \dfrac{\mathbf{v} \times \mathbf{a}}{v^2} \times (\mathbf{a} - \mathbf{g}) + T(\mathbf{v}, \mathbf{a})\mathbf{u} \\ -\gamma\mathbf{u} \end{bmatrix} x + \begin{bmatrix} 0 \\ 0 \\ 0 \\ I_{3\times3} \end{bmatrix} \mathbf{w}_b \tag{141}$$

where $\mathbf{p}$, $\mathbf{v}$, $\mathbf{a}$ are the position, velocity, and acceleration vectors, respectively, in an inertial frame and $T(\mathbf{v}, \mathbf{a}) = [\mathbf{t},\ \mathbf{b},\ \mathbf{n}]$ is the coordinate transformation matrix from the target body frame to the inertial frame, where $\mathbf{t} = \mathbf{v}/v$, $\mathbf{b} = (\mathbf{a} - \mathbf{g}) \times \mathbf{v}/\|(\mathbf{a} - \mathbf{g}) \times \mathbf{v}\|$, $\mathbf{n} = \mathbf{b} \times \mathbf{t}$ are the unit tangential, normal, and binormal vectors, respectively.

In this model, the (unperturbed) motions in the three spatial directions are coupled and confined to the maneuver plane, but not necessarily at a constant speed. The time-correlated perturbation $\mathbf{u}$ accounts for the (mostly out-of-plane) unmodeled motions and is modeled in the body frame to be more precise. This, however, necessitates a coordinate transformation. This in turn introduces a dependency of the model on velocity and acceleration estimates, which may lead to a model error and thus accuracy degradation. This generalized coordinated turn model does not estimate the turn rate explicitly and is highly nonlinear.

While the constant-turn models of Section VIB and the coordinated-turn models of Section VIC are similar, they are based on two quite different ideas. The former are of kinematic type, which aim at fitting typical target trajectories during a maneuver, while the development of the latter models relies on flight dynamics explicitly.

The generalized coordinated turn model (139) is more convenient than the coordinated turn model (e.g., without the need to determine the thrust and lift). However, the planar VT model (122) is even more convenient for most applications. The underlying assumptions for the constant-turn models of Section VIB are that (105) holds and that the turn rate or the angular velocity vector is constant. Note that it follows from (107) and $\dot{\mathbf{\Omega}} = 0$ that $\dot{\mathbf{a}} = \mathbf{\Omega} \times \mathbf{a}$. It differs from the generalized coordinated turn model only in the lack of the $\mathbf{\Omega} \times \mathbf{g}$ term, which is constant if $\mathbf{\Omega}$ is constant. In particular, the two models coincide when the maneuver plane is horizontal.

Some other 3D target maneuver models are hard to be grouped into the above classes. For example, the maneuver model $\ddot{b} = 0$, $\ddot{e} = 0$ was proposed in [97] with a state $x = [\mathsf{x}, \mathsf{y}, \mathsf{z}, v, b, \dot{b}, e, \dot{e}]'$, where $(\mathsf{x}, \mathsf{y}, \mathsf{z})$ and $v$ are target position and speed, respectively, and $b$ and $e$ are the bearing and elevation of the velocity vector, respectively.

## D. Flight Dynamics Based Models

An analysis of real trajectory data presented in [99] reveals that a target motion is not necessarily confined to a plane in reality, but all the above models are exclusively or primarily for planar motions and may be inadequate for more sophisticated target motions. Models that incorporate sufficient out-of-plane motions may be more desirable in such situations. A class of more sophisticated models, based on flight dynamics, appears to be a better alternative in this regard. The price is, however, a dramatic increase in nonlinearity, state dimension, and number of design parameters largely uncertain in a noncooperative tracking environment. Also, they are not really point-target models. We review these models next.

The basic idea of these models is to enhance the modeling accuracy by not only exploring more detailed flight-dynamics relationships but also explicitly accounting for the aircraft's controls. Were the controls known to trackers, these models would describe an aircraft motion more accurately than other models. So a fundamental issue here is the uncertainty in the pilot's control actions in a noncooperative environment. Modeling these controls as random processes is again a "standard" means to account for this uncertainty. We consider the model of [105] to illustrate features of these models.

The basis for this model is the rigid-body flight dynamics model of aircraft motion [83], given in the form

$$\dot{\mathbf{p}} = \mathbf{v} \tag{142}$$

$$\dot{\mathbf{v}} = (\mathbf{c} + \mathbf{c}_\alpha \alpha + \mathbf{c}_\beta \beta + C_\delta \boldsymbol{\delta}) v^2 + \mathbf{c}_T T + \mathbf{g} \tag{143}$$

$$\dot{\boldsymbol{\Omega}} = \mathbf{f}(\boldsymbol{\Omega}) + (\mathbf{d}_\alpha \alpha + \mathbf{d}_\beta \beta + D_\Omega \boldsymbol{\Omega} + D_\delta \boldsymbol{\delta}) v^2 + \mathbf{d}_T T \tag{144}$$

$$\dot{\boldsymbol{\Phi}} = G(\boldsymbol{\Phi})\boldsymbol{\Omega} \tag{145}$$

with

$$\mathbf{f}(\boldsymbol{\Omega}) = \begin{bmatrix} -(I_{xx} - I_{yy})qr/I_{xx} \\ -(I_{xx} - I_{zz})rp/I_{xx} \\ -(I_{yy} - I_{xx})pq/I_{xx} \end{bmatrix}$$

$$G(\boldsymbol{\Phi}) = \begin{bmatrix} 1 & \sin\phi\tan\theta & \cos\phi\tan\theta \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi\cos^{-1}\theta & \cos\phi\cos^{-1}\theta \end{bmatrix} \tag{146}$$

where $\mathbf{p} = [\mathsf{x},\mathsf{y},\mathsf{z}]'$ is the position in the inertial frame, $\boldsymbol{\Omega} = [p,q,r]'$ the angular velocity in the body frame, $\alpha$ the angle of attack, $\beta$ the sideslip, $\boldsymbol{\delta} = [\delta_A, \delta_R, \delta_E]'$ the angles of aileron, rudder, and elevator deflections, respectively, $\boldsymbol{\Phi} = [\psi,\theta,\phi]'$ an attitude vector of the Euler angles: roll (bank angle) $\psi$, pitch $\theta$, yaw (heading) $\phi$, $T$ the thrust, $\mathbf{g}$ the gravity, $I$ the moment of inertia, and $\mathbf{c}$, $\mathbf{c}_\alpha$, $\mathbf{c}_\beta$, $C_\delta$, $\mathbf{c}_T$, $\mathbf{d}_\alpha$, $\mathbf{d}_\beta$, $D_\delta$, $\mathbf{d}_T$ are coefficients. Equations (142)–(143) describe the aircraft body translational motion in the inertial frame, (144) is a moment equation describing the angular acceleration in the body frame, and (145) provides the relationship between the Euler angles and the angular velocity. The aircraft body rotational motion ((144)–(145)) strongly affects the translational motion ((142)–(143)) through the dependence of the model coefficients $\mathbf{c}(\boldsymbol{\Phi})$, $\mathbf{c}_\alpha(\boldsymbol{\Phi})$, $\mathbf{c}_\beta(\boldsymbol{\Phi})$, $C_\delta(\boldsymbol{\Phi})$, $\mathbf{c}_T(\boldsymbol{\Phi})$ on the attitude angles $\boldsymbol{\Phi}$ [105].

Given all the model coefficients, the uncertainty of a maneuver boils down to that of the control input: thrust $T$ and control angles $\boldsymbol{\delta}$. A state-space model can be developed in a trivial manner. For example, augment the state vector by $T$ and $\boldsymbol{\delta}$ and model their variations as zero-mean white noise processes [105]. The resulting state vector $x = [\mathbf{p}', \mathbf{v}', \boldsymbol{\Omega}', \boldsymbol{\Phi}', \boldsymbol{\delta}', T]'$ is 16-dimensional and the state-space model is given by (142)–(145), with white noise added to (143) and (144), plus

$$\dot{T} = w_T, \qquad \dot{\boldsymbol{\delta}} = \mathbf{w}_\delta. \tag{147}$$

The model (142)–(147) is quite general. What is crucial here is the determination of the model coefficients $\mathbf{c}(\boldsymbol{\Phi})$, $\mathbf{c}_\alpha(\boldsymbol{\Phi})$, $\mathbf{c}_\beta(\boldsymbol{\Phi})$, $C_\delta(\boldsymbol{\Phi})$, $\mathbf{c}_T(\boldsymbol{\Phi})$ and $\mathbf{d}_\alpha$, $\mathbf{d}_\beta$, $D_\Omega$, $D_\delta$, $\mathbf{d}_T$. This was done in [105] heuristically based on an analysis of a particular real trajectory of a given aircraft. Caution has to be exercised in the application of such results to other situations.

A more rigorous model, albeit not general, was proposed in [106]. Under some simplifying assumptions essentially the same as those for the coordinated turn, an analytical expression of the lift as a function of the attitude angles $\boldsymbol{\Phi}$ was derived, which explicitly describes the dependence of the acceleration $\dot{\mathbf{v}}$ on $\boldsymbol{\Phi}$ in (143). The effects of the zero-mean thrust (minus drag) term $T - D$ and deflection angles were collectively modeled as a first-order Markov (Singer) process $\gamma$, governed by $\dot{\gamma} = -(1/\tau)\gamma + \mathbf{w}_\gamma$, in place of (147) as above. The state vector $x = [\mathbf{p}', \mathbf{v}', \boldsymbol{\Omega}', \boldsymbol{\Phi}', \gamma']'$ is 15-dimensional.

In general, prediction accuracy of these models for a target state can be dramatically improved by accounting for the impact of the target attitude angles on the target acceleration. In fact, these models are clearly unobservable under position-only measurements. Estimating the attitude angles is possible if, in addition to the standard position (and possibly velocity) measurements, direct attitude measurements are available. The use of attitude measurements was first suggested in [43] and further developed in [107, 106, 108, 105], and others.

## VII. CONCLUDING REMARKS

The target maneuver models developed for target tracking can be classified into 1D, 2D, and 3D categories, according to the coupling between motions along different coordinates.

The 1D models have no (or weak) coupling among coordinates. Most of them are based on modeling the driving force (usually acceleration) of the target maneuver as a random process without recourse to the actual target dynamics or kinematics directly. These models form three families, corresponding to three classes of random processes: white noise, Markov processes, and semi-Markov jump processes. The simplest white-noise family models the target position derivative (or difference) of a certain order as white noise. It includes the classical CV and CA models. In the simple and widely used family of Markov processes, the target acceleration or jerk is modeled as a Markov process of various degrees of complexity. The most well-known representative of this family is the Singer model. The semi-Markov jump process models are the most sophisticated family of the three. It has quite good potential, but the intricacy involved in these models makes it hard for the practitioners to apply them.

The 2D and 3D models differ from the 1D models by not only their correlation across coordinates, but also their explicit dependence on the target kinematics and possibly dynamics. These models try to capture the important characteristics of the target behavior during maneuvers, which typically involve various turning motions. Many versions of (nearly) CT models have been developed. Few models are available that are valid for other maneuver motions. Apart from the popular CT models, it seems that the 2D approximate curvilinear motion model and the planar variable-turn model of (122) deserve more attention than they have received due to their attractive simplicity, applicability, and flexibility.

More accurate models are intuitively appealing, but they also pose problems and challenges. For example, they are usually highly nonlinear and thus require effective nonlinear filtering algorithms; they may lead to poor tracking performance due to their dependence on target state in the case of inaccurate state estimates, particularly for agile targets at a low data rate. That is partly why simple but less accurate models have their values and reasons to exist. While the choice of a particular model depends certainly on the particular application, a good understanding of pros and cons of each model is nonetheless extremely helpful.

Most of the models developed in the literature, especially 2D and 3D models, are particularly suitable for aircraft, although they are more or less applicable to many other targets. Many of these aircraft models are fairly accurate, but they have a common omission—wind effect is not considered explicitly. A reasonably large number of dynamic models have also been developed for ballistic targets (e.g., missiles), which will be covered separately in a subsequent part of this survey, of which [14] is a preliminary version. To our knowledge, however, few models have been developed that are particularly suitable for other targets, such as submarines, ships, and ground targets. This deserves more effort.


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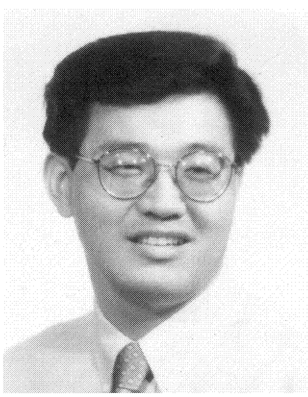

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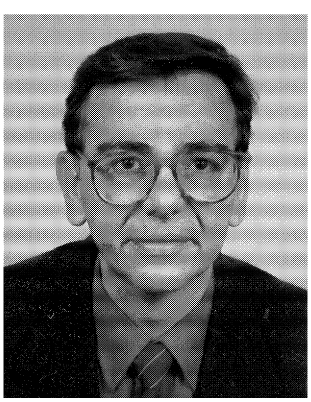

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