**IST, Digital Signal Processing, Test #2 and Exam #1 solution**
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**1.a)** From the table of ZT pairs and using the linearity of the ZT,

$$X(z) = \frac{5}{1-\frac{1}{2}z^{-1}}\,, \quad |z| > \frac{1}{2}\,, \qquad Y(z) = \underbrace{\frac{1}{1-\frac{1}{2}z^{-1}}}_{|z|>\frac{1}{2}} + \underbrace{\frac{4}{1+2z^{-1}}}_{|z|>2} = \frac{5}{\left(1-\frac{1}{2}z^{-1}\right)\left(1+2z^{-1}\right)}\,, \quad |z| > 2\,.$$

$$H(z) = \frac{Y(z)}{X(z)} = \frac{1}{1+2z^{-1}}\,, \quad |z| > 2 \quad \left(\begin{array}{l}\text{the ROC is such that, when intersected with the}\\ \text{ROC of } X(z)\text{, results in the ROC of } Y(z)\,.\end{array}\right)$$

**1.b)** The ROC of the system function does not include the unit-circle $\Longrightarrow$ its impulse response is not absolutely summable $\Longrightarrow$ the system is unstable.

**1.c)** $Y(z) = X(z)H(z) \Longleftrightarrow Y(Z) = X(z)/(1+2z^{-1}) \Longleftrightarrow Y(z) + 2z^{-1}Y(z) = X(z)$. Computing the inverse ZT and using its linearity and shifting properties, $y[n] + 2y[n-1] = x[n]$.

**2.a)** From the definition of 20-point DFT,

$$Y[k] = \sum_{n=0}^{19} y[n]e^{-j\frac{2\pi}{20}kn} = \sum_{n=0}^{19} y[n]e^{-j\frac{\pi}{10}kn}\,.$$

Since $Y[k] = e^{-j\frac{\pi}{10}k3}$, we immediately see that $y[3] = 1$ and $y[n] = 0$ for $n \neq 3$, *i.e.*, $y[n] = \delta[n-3]$.

**2.b)** $X[k]Y[k]$ is the DFT of the 20-point circular convolution of $x[n]$ and $y[n]$. Since $x[n]$ and $y[n]$ are of length, at most, 10, their 20-point circular convolution coincides with the linear convolution (because this is of length, at most, $10 + 10 - 1 = 19 < 20$). Thus, the signal with DFT $X[k]Y[k]$ is

$$x[n] * y[n] = x[n] * \delta[n-3] = x[n-3] = \begin{cases} 2 & \text{if } 3 \leq n \leq 12 \\ 0 & \text{otherwise}\,. \end{cases}$$

**2.c)** The signal with DFT $X[k]Y[k]$ is the 10-point circular convolution of $x[n]$ and $y[n]$:

$$\sum_{k=0}^{9} y[k]x[((n-k))_9] = \sum_{k=0}^{9} \delta[k-3]x[((n-k))_9] = x[((n-3))_9] = 2\,.$$

**3.a)** As requested by the sampling theorem, the sampling frequency should be larger than twice the highest frequency of the continuous signal, *i.e.*, $f_s > 2$KHz, *e.g.*, $f_s = 10$KHz $\Longleftrightarrow T = 1/f_s = 10^{-4}$s.

**3.b)** $X[n,0] = 0 \Longleftrightarrow \sum_{m=0}^{99} x[n+m] = 0$, which means the signal $x[n]$ is zero mean (when averaged within any 100-point window). As a consequence, the DC component of the signal $x_c(t)$ is also zero (within any window of size $100T = 0.01$s).

**3.c)** $X[400,k] = \sum_{m=0}^{99} x[400+m]e^{-j\frac{2\pi}{100}km}$ is the 100-point DFT of $y[n] = x[400+n], 0 \leq n \leq 99$. Thus,

$$\begin{aligned} y[n] &= \frac{1}{100}\sum_{k=0}^{99} Y[k]e^{j\frac{2\pi}{100}kn} = \frac{1}{100}\sum_{k=0}^{99} X[400,k]e^{j\frac{2\pi}{100}kn} = \frac{1}{100}\sum_{k=0}^{99} \left(50\delta[k-10] + 50\delta[k-90]\right)e^{j\frac{2\pi}{100}kn} \\ &= \frac{1}{2}e^{j\frac{2\pi}{100}10n} + \frac{1}{2}e^{j\frac{2\pi}{100}90n} = \frac{1}{2}e^{j\frac{\pi}{5}n} + \frac{1}{2}e^{j\frac{9\pi}{5}n} = \frac{1}{2}e^{j\frac{\pi}{5}n} + \frac{1}{2}e^{-j\frac{\pi}{5}n} = \cos\left(\frac{\pi}{5}n\right) = x[400+n]\,. \end{aligned}$$

We conclude that $x[n]$ is sinusoidal, with frequency $\omega = \pi/5$, for $400 \leq n \leq 499$. Thus, for $400T \leq t \leq 499T \Longleftrightarrow 0.04\text{s} \leq t \leq 0.05\text{s}$, $x_c(t)$ is also sinusoidal, with angular frequency $\Omega = \omega/T = 2\pi \times 10^3$ rad s$^{-1}$, *i.e.*, with frequency $\Omega/2\pi = 1$KHz.

**4.a)** Since $w$ is WGN, the 8 samples are independent, thus

$$p(\mathbf{x};A)=\prod_{n=0}^{7}p(x[n];A)=\prod_{n=0}^{7}\frac{1}{\sqrt{2\pi}}e^{-\frac{1}{2}\left(x[n]-A\sin\left(\frac{\pi}{4}n\right)\right)^2}=\frac{1}{(2\pi)^4}e^{-\frac{1}{2}\sum_{n=0}^{7}\left(x[n]-A\sin\left(\frac{\pi}{4}n\right)\right)^2},$$

$$\ln p(\mathbf{x};A)=-4\ln 2\pi-\frac{1}{2}\sum_{n=0}^{7}\left(x[n]-A\sin\left(\frac{\pi}{4}n\right)\right)^2,\qquad \frac{\partial \ln p(\mathbf{x};A)}{\partial A}=\sum_{n=0}^{7}\sin\left(\frac{\pi}{4}n\right)\left(x[n]-A\sin\left(\frac{\pi}{4}n\right)\right),$$

$$\frac{\partial^2\ln p}{\partial A^2}=-\sum_{n=0}^{7}\sin^2\!\left(\frac{\pi}{4}n\right)=-\left(0+\frac{1}{2}+1+\frac{1}{2}+0+\frac{1}{2}+1+\frac{1}{2}\right)=-4,\quad \mathrm{CRB}(A)=\frac{1}{-\mathrm{E}\left\{\frac{\partial^2\ln p(\mathbf{x};A)}{\partial A^2}\right\}}=\frac{1}{4}.$$

**4.b)** $\mathrm{E}\left\{\hat{A}\right\}=\frac{1}{2}\mathrm{E}\{x[2]-x[6]\}=\frac{1}{2}\mathrm{E}\{A1+w[2]-A(-1)-w[6]\}=\frac{1}{2}\left(2A+\mathrm{E}\{w[2]\}-\mathrm{E}\{w[6]\}\right)=A\Leftrightarrow\hat{A}$ is unbiased.

**4.c)** $\mathrm{Var}\left\{\hat{A}\right\}=\mathrm{Var}\left\{\frac{1}{2}\left(x[2]-x[6]\right)\right\}=\frac{1}{4}\mathrm{Var}\left\{2A+w[2]-w[6]\right\}=\frac{1}{4}\left(\mathrm{Var}\left\{2A\right\}+\mathrm{Var}\left\{w[2]\right\}+\mathrm{Var}\left\{w[6]\right\}\right)=$
$=\frac{1}{4}(0+1+1)=\frac{1}{2}>\mathrm{CRB}(A)\implies\hat{A}$ is not efficient.

**5.a)** Since $w$ is WGN, the $2N+1$ observations are independent, thus, as in **4.a)**, we have

$$p(\mathbf{x};A,B)=\frac{1}{\left(2\pi\sigma^2\right)^{\frac{2N+1}{2}}}e^{-\frac{1}{2\sigma^2}\sum_n(x[n]-An-B)^2},\quad \ln p(\mathbf{x};A,B)=\text{const.}-\frac{1}{2\sigma^2}\sum_{n=-N}^{N}\left(x[n]-An-B\right)^2.$$

$$\frac{\partial\ln p(\mathbf{x};A,B)}{\partial A}=\sum_{n=-N}^{N}n\left(x[n]-An-B\right)=\sum_{n=-N}^{N}nx[n]-A\sum_{n=-N}^{N}n^2-B\sum_{n=-N}^{N}n=\sum_{n=-N}^{N}nx[n]-A\sum_{n=-N}^{N}n^2,$$

$$\frac{\partial\ln p(\mathbf{x};A,B)}{\partial B}=\sum_{n=-N}^{N}\left(x[n]-An-B\right)=\sum_{n=-N}^{N}x[n]-A\sum_{n=-N}^{N}n-B\sum_{n=-N}^{N}1=\sum_{n=-N}^{N}x[n]-(2N+1)B.$$

$$\begin{cases}\dfrac{\partial\ln p(\mathbf{x};A,B)}{\partial A}=0\\[2ex]\dfrac{\partial\ln p(\mathbf{x};A,B)}{\partial B}=0\end{cases}\iff\begin{cases}\hat{A}_{\mathrm{ML}}=\dfrac{\sum_{n=-N}^{N}nx[n]}{\sum_{n=-N}^{N}n^2}\\[2ex]\hat{B}_{\mathrm{ML}}=\dfrac{\sum_{n=-N}^{N}x[n]}{2N+1}\end{cases}$$

**5.b)** The ML estimate is asymptotically unbiased and efficient, which means that, when the number of independent observations grows, its expected value approaches the true value of the parameter and its variance approaches the CRB. This is case of this problem because the observations are independent. Thus, although this may also happen for finite $N$, we have at least the guarantee that, when $N\to\infty$,

$$\mathrm{E}\left\{\hat{A}_{\mathrm{ML}}\right\}\to A,\quad \mathrm{E}\left\{\hat{B}_{\mathrm{ML}}\right\}\to B,\quad \mathrm{Var}\left\{\hat{A}_{\mathrm{ML}}\right\}\to\mathrm{CRB}(A),\quad \mathrm{Var}\left\{\hat{B}_{\mathrm{ML}}\right\}\to\mathrm{CRB}(B).$$

**5.c)** According to the ML invariance property, the ML estimate of a function of a parameter vector is just the function evaluated at the ML estimate of the parameter vector. Thus, the estimate of the noiseless signal at $n=-1$, *i.e.*, of $C=B-A$, is

$$\hat{C}_{\mathrm{ML}}=\hat{B}_{\mathrm{ML}}-\hat{A}_{\mathrm{ML}}=\frac{0+6+6}{2\times1+1}-\frac{(-1)\times0+0\times6+1\times6}{(-1)^2+0^2+1^2}=1.$$

**6.a)** Using the Bayes law for conditional densities,

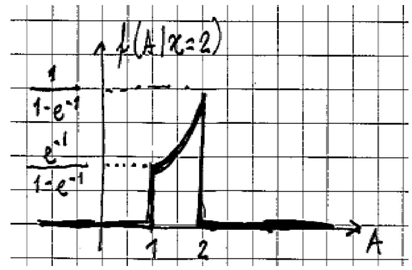
$$p(A|x)=\frac{p(A,x)}{p(x)}=\frac{p(A,x)}{\int_{-\infty}^{+\infty}p(A,x)\,dA}=\frac{p(x|A)p(A)}{\int_{-\infty}^{+\infty}p(x|A)p(A)\,dA}.$$

$x=A+w$. Given $A$, $x$ has the same distribution of $w$, with a different mean, $p(x|A)=e^{-(x-A)}u(x-A)$.

$$p(A)=\begin{cases}1/2 & \text{if } 1\le A\le 3;\\ 0 & \text{otherwise},\end{cases}\qquad p(x=2|A)p(A)=e^{-(2-A)}u(2-A)p(A)=\begin{cases}e^{A-2}/2 & \text{if } 1\le A\le 2;\\ 0 & \text{otherwise}.\end{cases}$$

$$\int_{-\infty}^{+\infty}p(x=2|A)p(A)\,dA=\int_{1}^{2}\frac{e^{A-2}}{2}\,dA=\frac{e^{-2}}{2}\int_{1}^{2}e^{A}\,dA=\frac{e^{-2}}{2}\left(e^{A}\right)\Big|_{1}^{2}=\frac{e^{-2}}{2}\left(e^{2}-e^{1}\right)=\frac{1-e^{-1}}{2}.$$

$$p(A|x=2) = \frac{p(x=2|A)p(A)}{\int p(x=2|A)p(A)\,dA} = \begin{cases} e^{A-2}/(1-e^{-1}) & \text{if } 1 \le A \le 2; \\ 0 & \text{otherwise}. \end{cases}$$




**6.b)** $\hat{A}_{\mathrm{MMSE}} = \mathrm{E}\{A|x=2\} = \int_{-\infty}^{+\infty} Ap(A|x=2)\,dA = \int_1^2 A\frac{e^{A-2}}{1-e^{-1}}\,dA$

$$= \frac{e^{-2}}{1-e^{-1}}\int_1^2 Ae^A\,dA = \frac{e^{-2}}{1-e^{-1}}\left(Ae^A - e^A\right)\Big|_1^2$$

$$= \frac{e^{-2}}{1-e^{-1}}\left(2e^2 - e^2 - 1e^1 + e\right) = \frac{1}{1-e^{-1}} \simeq 1.58$$

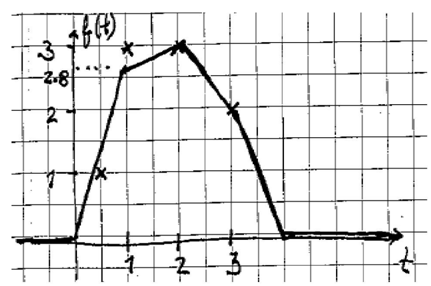
**6.c)** $\hat{A}_{\mathrm{MAP}} = \arg\max_A p(A|x=2) = 2$, as it is clear from the expression of $p(A|x=2)$ or its sketch in **6.a)**. The MAP estimate is optimal in the sense of minimizing the risk given by the expected value of the hit-or-miss cost, which penalizes erroneous estimates with a positive constant, *i.e.*, independent of the magnitude of the error.

**7.a)** The model $x_i \simeq c_1\phi_1(t_i) + c_2\phi_2(t_i) + c_3\phi_3(t_i)$ is $\mathbf{x} \simeq \mathbf{\Phi c}$, with

$$\mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ 3 \\ 2 \end{bmatrix}, \quad \mathbf{\Phi} = \begin{bmatrix} \phi_1(0.5) & \phi_2(0.5) & \phi_3(0.5) \\ \phi_1(1) & \phi_2(1) & \phi_3(1) \\ \phi_1(2) & \phi_2(2) & \phi_3(2) \\ \phi_1(3) & \phi_2(3) & \phi_3(3) \end{bmatrix} = \begin{bmatrix} 0.5 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \quad \text{and} \quad \mathbf{c} = \begin{bmatrix} c_1 \\ c_2 \\ c_3 \end{bmatrix}.$$

$\hat{\mathbf{c}}_{\mathrm{LS}} = \arg\min_{\mathbf{c}}(\mathbf{x}-\mathbf{\Phi c})^T(\mathbf{x}-\mathbf{\Phi c}) = (\mathbf{\Phi}^T\mathbf{\Phi})^{-1}\mathbf{\Phi}^T\mathbf{x}$ (which exists and is unique if and only if $\mathbf{\Phi}^T\mathbf{\Phi}$ is nonsingular).

$$\mathbf{\Phi}^T\mathbf{\Phi} = \begin{bmatrix} 1.25 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{bmatrix}, \ \Phi^T\mathbf{x} = \begin{bmatrix} 3.5 \\ 3 \\ 2 \end{bmatrix}, \ \hat{\mathbf{c}}_{\mathrm{LS}} = \begin{bmatrix} 3.5/1.25 \\ 3 \\ 2 \end{bmatrix} = \begin{bmatrix} 2.8 \\ 3 \\ 2 \end{bmatrix}.$$




To sketch $f(t) = c_1\phi(t-1) + c_2\phi(t-2) + c_3\phi(t-3)$, we just note that $f$ is continuous, $f(1) = c_1$, $f(2) = c_2$, $f(3) = c_3$, $f(n) = 0$ for other integers $n$, and, for $n < t < n+1$, $f(t)$ is linear. The total approximation error is $\epsilon^2 = (1-1.4)^2 + (3-2.8)^2 + (3-3)^2 + (2-2)^2 = 0.4^2 + 0.2^2 = 0.2$.

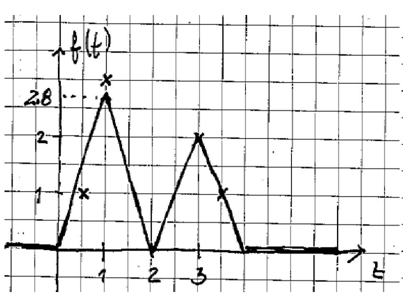
**7.b)**

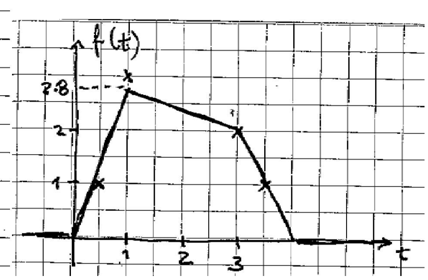
$$\mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ 2 \\ 1 \end{bmatrix}, \quad \mathbf{\Phi} = \begin{bmatrix} 0.5 & 0 & 0 \\ 1 & 0 & 0 \\ 0 & 0 & 1 \\ 0 & 0 & 0.5 \end{bmatrix}, \quad \mathbf{\Phi}^T\mathbf{\Phi} = \begin{bmatrix} 1.25 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 1.25 \end{bmatrix}, \ \mathbf{\Phi}^T\mathbf{\Phi} \text{ is singular} \implies \hat{\mathbf{c}}_{\mathrm{LS}} \text{ is not unique.}$$

$\mathbf{x} \simeq \mathbf{\Phi c} = \widetilde{\mathbf{\Phi}}\begin{bmatrix} c_1 \\ c_3 \end{bmatrix} \Longrightarrow \begin{bmatrix} \hat{c}_{1\mathrm{LS}} \\ \hat{c}_{3\mathrm{LS}} \end{bmatrix} = (\widetilde{\mathbf{\Phi}}^T\widetilde{\mathbf{\Phi}})^{-1}\widetilde{\mathbf{\Phi}}^T\mathbf{x}$ and $\hat{c}_{2\mathrm{LS}}$ is arbitrary. $\widetilde{\mathbf{\Phi}}$ contains 1st and 3rd columns of $\mathbf{\Phi}$,

$\widetilde{\mathbf{\Phi}}^T\widetilde{\mathbf{\Phi}} = \begin{bmatrix} 1.25 & 0 \\ 0 & 1.25 \end{bmatrix}$, $\widetilde{\Phi}^T\mathbf{x} = \begin{bmatrix} 3.5 \\ 2.5 \end{bmatrix}$,

$\begin{bmatrix} \hat{c}_{1\mathrm{LS}} \\ \hat{c}_{3\mathrm{LS}} \end{bmatrix} = \begin{bmatrix} 3.5/1.25 \\ 2.5/1.25 \end{bmatrix} = \begin{bmatrix} 2.8 \\ 2 \end{bmatrix}$.




Sketches for $\hat{c}_{2\mathrm{LS}} = 0$ and $\hat{c}_{2\mathrm{LS}} = 2.4$. The error is $\epsilon^2 = 0.2$ for any solution.

**7.c)**

$$\mathbf{x} = \begin{bmatrix} 1 \\ 3 \\ 2 \end{bmatrix}, \quad \mathbf{\Phi} = \begin{bmatrix} 0.5 & 0 & 0 \\ 0 & 0.5 & 0.5 \\ 0 & 0 & 1 \end{bmatrix}.$$

The model is exact ($\epsilon^2 = 0$) and with an unique solution:

$$\mathbf{x} = \mathbf{\Phi c} \iff \begin{cases} 1 = 0.5c_1 \\ 3 = 0.5c_1 + 0.5c_2 \\ 2 = c_3 \end{cases} \iff \begin{cases} c_1 = 2 \\ c_2 = 4 \\ c_3 = 2. \end{cases}$$