INSTITUTO SUPERIOR TÉCNICO

DSP 10/11 — Digital Signal Processing, 2nd Test and 1st Exam, June 3rd, 2011

Test: problems **4** to **7**. Duration: 2 hours. Exam: all problems except **5**. Duration: 3 hours. Show all your work on the exam pages and make sure you justify all your answers (results that are not explained or justified may count less, even if they are correct). Good luck!

1. Consider the LTI system that responds to the input $5\left(\frac{1}{2}\right)^n u[n]$ with the output $\left(\left(\frac{1}{2}\right)^n + 4(-2)^n\right) u[n]$.

a) Find its system function, $H(z)$, and corresponding region of convergence.

b) Is the system stable?

c) Find a difference equation satisfied by any pair input $x[n]$ / output $y[n]$.

2. Consider two finite-length signals, $x[n]$ and $y[n]$, that are zero outside the range $0 \le n \le 9$. We know the values of $x[n]$ and the 20-point DFT of $y[n]$, $0 \le n \le 19$:

$$x[n] = 2\,, \quad 0 \le n \le 9\,, \qquad\qquad Y[k] = e^{-j\frac{3\pi}{10}k}\,, \quad 0 \le k \le 19\,.$$

a) Find $y[n]$.

b) Find the signal with DFT given by $X[k]Y[k]$ (20-point DFTs).

c) Repeat the item above, now with 10-point DFTs.

3. With the goal of analyzing a bandlimited continuous-time signal $x_c(t)$, whose highest frequency is 1KHz, we sample it, obtaining $x[n] = x_c(nT)$. Then, we compute the localized FT of $x[n]$, *i.e.*, its spectrogram, as given by

$$X[n,k] = \sum_{m=0}^{99} x[n+m] e^{-j\frac{2\pi}{100}km}\,, \qquad 0 \le k \le 99\,.$$

a) Find an appropriate value for T.

b) Given that $X[n,0] = 0$, what can we say about $x_c(t)$?

c) Given that $X[400,k] = 50\delta[k-10] + 50\delta[k-90]$, what can we say about $x_c(t)$?

4. To estimate the parameter A from the observation of a signal

$$x[n] = A \sin\left(\frac{\pi}{4} n\right) + w[n]\,, \quad 0 \le n \le 7\,,$$

where w is zero mean white Gaussian noise (WGN) with unitary variance, one proposes the estimator

$$\hat{A} = \frac{1}{2}\left(x[2] - x[6]\right)\,.$$

a) Find the Cramer-Rao bound (CRB) for the estimation of A.

b) Is the estimator $\hat{A}$ unbiased?

c) Is the estimator $\hat{A}$ efficient?

5. Consider noisy observations $x[n] = An + B + w[n], -N \le n \le N$, where w is zero mean WGN with variance σ^2.

a) Find expressions for the maximum likelihood (ML) estimates of A e B.

b) What can we say about these estimates when $N \to \infty$?

c) For $N = 1$ and $x[-1] = 0$, $x[0] = x[1] = 6$, find the ML estimate of the value of the noiseless signal at $n = -1$, *i.e.*, the ML estimate of a parameter $C = B - A$.

6. Consider the observation model $x = A + w$, where the noise w follows an exponential distribution, $p(w) = e^{-w}u(w)$, and A is a random variable uniformly distributed between 1 and 3. We observed $x = 2$.

a) Find and sketch the *a posteriori* probability density function of A, $p(A|x = 2)$.

b) Find the minimum mean square error (MMSE) estimate of A. (a primitive of xe^x is $xe^x - e^x$)

c) Find the maximum *a posteriori* (MAP) estimate of A. Under which criterion is this estimate optimal?

7. We want to approximate the set of points $\{(t_i, x_i), i = 1, \ldots, N\}$ by making $x_i \simeq f(t_i)$, where $f(t)$ is a linear combination of first-order splines, *i.e.*, $f(t) = \sum_{k=1}^{3} c_k \phi_k(t)$, with $\phi_k(t) = \phi(t-k)$ and

$$\phi(t) = \begin{cases} 1 - |t| & \text{if } |t| < 1; \\ 0 & \text{otherwise}\,. \end{cases}$$

For each of the following sets of points, find the coefficients c_1, c_2, and c_3, using the method of least squares (LS), sketch the resulting $f(t)$, and find the total approximation error, making clear if the solution is unique (if it is, justify; if not, find at least two distinct solutions).

a) $N = 4$ points, $\{(0.5, 1), (1, 3), (2, 3), (3, 2)\}$.

b) $N = 4$ points, $\{(0.5, 1), (1, 3), (3, 2), (3.5, 1)\}$.

c) $N = 3$ points, $\{(0.5, 1), (2.5, 3), (3, 2)\}$.