IST, Digital Signal Processing, Test #1 solution
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April, 2011

1.a) Since the frequency response is given by $H(e^{j\omega}) = Y(e^{j\omega})/X(e^{j\omega})$, where $Y(e^{j\omega})$ is the Fourier Transform (FT) of $y[n]$ and $X(e^{j\omega})$ is the FT of $x[n]$, we equate the FTs of both sides of the difference equation. Using the linearity of the FT and its time shifting property (the FT of $z[n-n_0]$ is $e^{-j\omega n_0}Z(e^{j\omega})$),

$$y[n] + \frac{1}{2}y[n-1] = 15x[n] \quad \xrightarrow{\text{FT}} \quad Y\left(e^{j\omega}\right)\left(1 + \frac{1}{2}e^{-j\omega}\right) = 15X\left(e^{j\omega}\right) \quad \Longleftrightarrow$$

$$\Longleftrightarrow \quad Y\left(e^{j\omega}\right)\left(1 + \frac{1}{2}e^{-j\omega}\right) = 15X\left(e^{j\omega}\right) \quad \Longleftrightarrow \quad \frac{Y\left(e^{j\omega}\right)}{X\left(e^{j\omega}\right)} = \frac{15}{1 + \frac{1}{2}e^{-j\omega}} = H\left(e^{j\omega}\right).$$

1.b) The impulse response $h[n]$ is the inverse FT of $H(e^{j\omega}) = 15/(1 + (1/2)e^{-j\omega})$. The FT of $a^n u[n]$, for $|a| < 1$, is $1/(1 - ae^{-j\omega})$ (from the table of FT pairs), thus we immediately get $h[n] = 15(-1/2)^n u[n]$.

1.c) For an LTI system with frequency response $H(e^{j\omega})$, if the input is of the form $x[n] = \cos \omega n$, the corresponding output is $y[n] = |H(e^{j\omega})| \cos(\omega n + \arg H(e^{j\omega}))$. In our case, $x[n] = \cos((\pi/2)n)$,

$$H\left(e^{j\frac{\pi}{2}}\right) = \frac{15}{1 + \frac{1}{2}e^{-j\frac{\pi}{2}}} = \frac{15}{1 - \frac{1}{2}j} = 6(2 + j), \quad \left|H\left(e^{j\frac{\pi}{2}}\right)\right| = 6\sqrt{5}, \quad \arg H\left(e^{j\frac{\pi}{2}}\right) = \arctan\frac{1}{2},$$

$$y[n] = \left|H\left(e^{j\frac{\pi}{2}}\right)\right| \cos\left(\frac{\pi}{2}n + \arg H\left(e^{j\frac{\pi}{2}}\right)\right) = 6\sqrt{5}\cos\left(\frac{\pi}{2}n + \arctan\frac{1}{2}\right).$$

2.a) The ROC of the system function extends outwords to infinity ($|z| > 2$) $\Longrightarrow$ the system impulse response $h[n]$ is right-sided $\Longrightarrow$ the system is causal ($h[n] = 0$ for $n < 0$ because, in addition, the number of zeros of $H(z)$ does not exceed its number of poles).

2.b) The ROC of the system function does not include the unit-circle $\Longrightarrow$ its impulse response is not absolutely summable $\Longrightarrow$ the system is unstable.

2.c) The z-Transform (ZT) of the output $y[n]$ is $Y(z) = X(z)H(z)$, where $X(z)$ is the ZT of $x[n] = u[n]$. From the table of ZT pairs, $X(z) = 1/(1 - z^{-1})$, $|z| > 1$, thus

$$Y(z) = \frac{5}{(1 - z^{-1})(1 - 2z^{-1})}, \quad \left\{\begin{array}{l} |z| > 1 \\ |z| > 2 \end{array}\right. \Longleftrightarrow |z| > 2.$$

$$Y(z) = \frac{5(-1)\left(-\frac{1}{2}\right)}{\left(z^{-1} - 1\right)\left(z^{-1} - \frac{1}{2}\right)} = \frac{A}{z^{-1} - 1} + \frac{B}{z^{-1} - \frac{1}{2}}, \qquad A = \frac{\frac{5}{2}}{1 - \frac{1}{2}} = 5, \quad B = \frac{\frac{5}{2}}{\frac{1}{2} - 1} = -5.$$

Selecting the ROCs of the two terms in such a way that their intersection is the ROC of $Y(z)$ ($|z| > 2$),

$$Y(z) = -5\underbrace{\frac{1}{1 - z^{-1}}}_{|z|>1} + 10\underbrace{\frac{1}{1 - 2z^{-1}}}_{|z|>2}.$$

The ZT of $a^n u[n]$ is $1/(1 + az^{-1})$, $|z| > |a|$ (from the table), thus, using the linearity of the ZT, we get

$$y[n] = -5u[n] + 10 \times 2^n u[n] = 5\left(-1 + 2^{n+1}\right)u[n].$$

3.a) Using the definition of DFT, the 10-point DFT of $x_1[n]$ is, for $0 \leq k \leq 9$,

$$X_1[k] = \sum_{n=0}^{9} x_1[n]e^{-j\frac{2\pi}{10}kn} = \sum_{n=0}^{9} \cos\left(\frac{3\pi}{5}n\right)e^{-j\frac{\pi}{5}kn} = \sum_{n=0}^{9} \frac{e^{j\frac{3\pi}{5}n} + e^{-j\frac{3\pi}{5}n}}{2}e^{-j\frac{\pi}{5}kn}$$

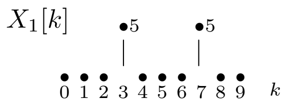
$$= \frac{1}{2}\sum_{n=0}^{9} e^{-j\frac{\pi}{5}(k-3)n} + \frac{1}{2}\sum_{n=0}^{9} e^{-j\frac{\pi}{5}(k+3)n} = \frac{1}{2}\sum_{n=0}^{9}\left(e^{-j\frac{\pi}{5}(k-3)}\right)^n + \frac{1}{2}\sum_{n=0}^{9}\left(e^{-j\frac{\pi}{5}(k+3)}\right)^n.$$

Let us evaluate each of the sums separately. If $k=3$, $\sum_{n=0}^{9}\left(e^{-j\frac{\pi}{5}(k-3)}\right)^n=\sum_{n=0}^{9}1=10$. If $k\neq 3$,

$$\sum_{n=0}^{9}\left(e^{-j\frac{\pi}{5}(k-3)}\right)^n=\frac{1-\left(e^{-j\frac{\pi}{5}(k-3)}\right)^{10}}{1-e^{-j\frac{\pi}{5}(k-3)}}=\frac{1-e^{-j\frac{\pi}{5}(k-3)10}}{1-e^{-j\frac{\pi}{5}(k-3)}}=\frac{1-e^{-j2\pi(k-3)}}{1-e^{-j\frac{\pi}{5}(k-3)}}=\frac{1-1}{1-e^{-j\frac{\pi}{5}(k-3)}}=0\,.$$

If $k{=}7$, $\sum_{n=0}^{9}\left(e^{-j\frac{\pi}{5}(k+3)}\right)^n=\sum_{n=0}^{9}\left(e^{-j2\pi}\right)^n=\sum_{n=0}^{9}1=10$. If $k{\neq}7$, $\sum_{n=0}^{9}\left(e^{-j\frac{\pi}{5}(k+3)}\right)^n=0$ as above.

$$X_1[k]=\begin{cases}5 & \text{if } k\in\{3,7\}\\ 0 & \text{if } k\in\{0,1,2,4,5,6,8,9\}\end{cases}$$



3.b) Since the N-point DFT of a length-N signal is given by the N equally spaced samples of its FT,

$$X_2[k]=X_2\left(e^{j\frac{2\pi}{10}k}\right)=X_2\left(e^{j\frac{\pi}{5}k}\right)=\frac{1-2^{10}e^{-j10\frac{\pi}{5}k}}{1-2e^{-j\frac{\pi}{5}k}}=\frac{1-2^{10}e^{-j2\pi k}}{1-2e^{-j\frac{\pi}{5}k}}=\frac{1-2^{10}}{1-2e^{-j\frac{\pi}{5}k}}\,,\quad 0\le k\le 9\,.$$

3.c) 5 samples of an FT would be enough to uniquely determine a length-5 signal. However, since the length $x_3[n]$ may be larger than 5 (we only know it is zero outside $0\le n\le 9$), we may have time-domain aliasing, thus the 5 samples of $X_3(e^{j\omega})$ do not uniquely determine $x_3[n]$; they only determine the length-5 aliased signal $\widetilde{x}_3[n]=\sum_{r=-\infty}^{+\infty}x_3[n+5r]$, $0\le n\le 4$. Since $x_3[n]$ is zero outside $0\le n\le 9$, we have $\widetilde{x}_3[n]=x_3[n]+x_3[n+5]$, $0\le n\le 4$. From the values of the 5 samples $X_3(e^{j\frac{2\pi}{5}k})=\widetilde{X}_3[k]=1$, we immediately get $\widetilde{x}_3[n]=\delta[n]$, $0\le n\le 4$ (the FT of $\delta[n]$ is 1, thus its DFT of arbitrary length is 1). Then, any solution for $x_3[n]$, $0\le n\le 9$, must only satisfy $\delta[n]=x_3[n]+x_3[n+5]$, $0\le n\le 4$. Two examples are $x_3^i[n]=\delta[n]$, $0\le n\le 9$, and $x_3^{ii}[n]=(1/2)\delta[n]+(1/2)\delta[n-5]$, $0\le n\le 9$.

4.a) The index $n=30000$ in $X[n,k]$ corresponds to the analysis of $x[l]$ in the time window $30000\le l\le 30049$. Choosing the starting point of this window as the reference for the time instant, the index $n=30000$ in $X[n,k]$ corresponds to the discrete-time instant 30000, thus to the continuous instant $t=30000T$, where $T=1/10^4=10^{-4}$ seg is the sampling period. This way, $t=30000\times 10^{-4}=3$ seg.

4.b) The index $k=50$ in $X[n,k]$ corresponds to the discrete-time frequency $\omega=(2\pi/500)50=\pi/5$, thus to the continuous-time angular frequency $\Omega=\omega/T=(\pi/5)/10^{-4}=2000\pi$ rad seg^{-1} and to the continuous-time frequency $f=\Omega/2\pi=1000$ seg$^{-1}=1000$ Hz.

4.c) The highest frequency of $x_c(t)$ is 1040 Hz and the sampling frequency is 10000 Hz, thus, since $10000>2\times 1040$, $x[n]$ uniquely determines $x_c(t)$ (Nyquist Sampling Theorem). The two components of $x_c(t)$ have frequencies separated by $\Delta_f=40\text{Hz}\Longleftrightarrow\Delta_\omega=2\pi T\Delta_f=2\pi 10^{-4}\times 40=\pi/125$. The spacing between consecutive frequency samples in the analysis is $2\pi/500=\pi/250$, which is smaller than Δ_ω, thus enough to discriminate between the frequencies of the two components of $x_c(t)$. What remains to be checked is the impact of the finite window of analysis, since the frequency resolution depends on that. From the table of FT pairs, the FT of the 50-point window used in the analysis is

$$w[n]=\begin{cases}1 & 0\le n\le 49\\ 0 & \text{otherwise}\end{cases}\quad\xrightarrow{\text{FT}}\quad W\big(e^{j\omega}\big)=\frac{\sin 25\omega}{\sin\frac{\omega}{2}}e^{-j\omega\frac{49}{2}}\,.$$

The width of the main lobe of $W\big(e^{j\omega}\big)$ is given by its smallest positive zero: $W\big(e^{j\omega}\big)=0\Longleftrightarrow\sin 25\omega=0\Longleftrightarrow 25\omega=\pi\Longleftrightarrow\omega=\pi/25$, which is greater than Δ_ω. As a consequence, the two components of $x_c(t)$ fall within the range of the main lobe of $W\big(e^{j\omega}\big)$ and the proposed analysis fails to identify them.