INSTITUTO SUPERIOR TÉCNICO

DSP 10/11 — Digital Signal Processing, 2$^{\text{nd}}$ Exam, June 25$^{\text{th}}$, 2011

Duration: 3 hours. Justify all your answers (results that are not explained or justified may count less, even if they are correct).

[4 val.]
1. Consider the causal LTI system for which the input x[n] and the output y[n] satisfy the difference equation

$$y[n] - 3y[n-1] = x[n] - x[n-1]\,.$$

a) Find its system function, $H(z)$, and corresponding region of convergence.

b) Find its impulse response, $h[n]$.

c) Is the system stable?

d) Find the output $y[n]$ when the input is $x[n] = 3(-3)^n u[n]$.

[3 val.]
2. Consider two finite-length signals, $x[n]$ and $y[n]$, that are zero outside the range $0 \leq n \leq 9$. We know the values of $x[n]$ and the Fourier Transform (FT) of $y[n]$:

$$x[n] = \begin{cases} 1 & \text{if } 0 \leq n \leq 9; \\ 0 & \text{otherwise}\,. \end{cases} \qquad , \qquad Y\left(e^{j\omega}\right) = 2e^{-j\omega} + e^{-3j\omega}\,.$$

a) Find $X[k]$, the 20-point DFT of $x[n]$.

b) Find $Y[k]$, the 20-point DFT of $y[n]$.

c) Find and sketch the signal with DFT given by $X[k]Y[k]$ (20-point DFTs).

[3 val.]
3. We want to find the impulse response $h[n]$ of a FIR system that approximates an ideal low-pass filter with cut-off frequency $\pi/4$, using the windowing method.

a) What is the impact of the choices of the type and dimension of the window?

b) Find the FIR filter, using a 3-point rectangular window, *i.e.*, find $h[0]$, $h[1]$, and $h[2]$.

c) Find the amplitude of the output of the FIR filter when the input is sinusoidal with frequency $\pi/2$ and amplitude 1. (if you did not solve b), consider $h[0] = 3$, $h[1] = -2$, and $h[2] = 3$.)

[6 val.]

4. Consider noisy observations of the sum of two sinusoidal signals,

$$x[n] = A\cos(\omega_a n) + B\cos(\omega_b n) + w[n], \quad 0 \leq n \leq N-1,$$

where the parameters A and B are unknown and w is zero mean white Gaussian noise (WGN) with variance σ^2. The frequencies of the signals are known: $\omega_a = 6\pi/N$ e $\omega_b = 10\pi/N$ (note that, in this case, we obtain $\sum_{n=0}^{N-1}\cos(\omega_a n)\cos(\omega_b n) = 0$ and $\sum_{n=0}^{N-1}\cos^2(\omega_a n) = \sum_{n=0}^{N-1}\cos^2(\omega_b n) = N/2$).

a) Find the Cramer-Rao bound (CRB) for the estimation of A and B.

b) Find expressions for the maximum likelihood (ML) estimates of A e B.

c) Are those estimates unbiased?

d) Are those estimates efficient?

e) If we knew the true value of B, could we estimate A with higher accuracy?

f) Repeat e), now for the case where the frequencies do not have the expressions above (to completely clarify your answer, use a particular case that results simple, *e.g.*, $N = 2$, $\omega_a = \pi/2$, and $\omega_b = \pi/3$).

[4 val.]

5. Consider the observation model $x = A + w$, where the noise w follows a Gaussian distribution with zero mean and unit variance and A is a random variable, independent from w, from which we know the *a priori* distribution.

a) Consider that A is *a priori* Gaussian with mean 4 and variance 2. We observed $x = 7$. Find the minimum mean square error (MMSE) estimate of A.

b) Now consider the that *a priori* A has mean 4 but it is uniformly distributed between 0 e 8. Sketch the *a posteriori* probability density function of A, $p(A|x)$, as a function of A, for fixed x (do not compute the normalization factors but present sketches for the values of x that originate qualitatively different plots), and find the maximum *a posteriori* (MAP) estimate of A as a function of x.