The World of Fourier and Wavelets:
Theory, Algorithms and Applications

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Cover photograph by Christiane Grimm, Geneva, Switzerland.

Experimental set up by Prof. Libero Zuppiroli, Laboratory of Optoelectronics Molecular Materials, EPFL, Lausanne, Switzerland.

The photograph captures an experiment first described by Isaac Newton in “Opticks” in 1730. Newton indicates how white light can be split into its color components and then resynthesized. It is a physical implementation of a decomposition into Fourier components, followed by a synthesis to recover the original, where the components are the colors of the rainbow. This experiment graphically summarizes the major theme of the book—many signals or functions can be split into essential components, from which the original can be recovered.
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Gabor Some information here.

0Last edit: JK Mar 04 08
natural numbers \( \mathbb{N} \) 0, 1, \ldots
integer numbers \( \mathbb{Z} \) \ldots, \(-1, 0, 1, \ldots
real numbers \( \mathbb{R} \) \((\mathbb{R}, \infty)\)
complex numbers \( \mathbb{C} \) \(a + jb, re^{j\theta}\)
a generic vector space \( V \) §1.2
a generic Hilbert space \( H \) §1.3
real part of \( \mathbb{R}(\cdot) \)
imaginary part of \( \mathbb{I}(\cdot) \)
closure of set \( S \)
functions \( x(t) \) argument \( t \) is continuous valued, \( t \in \mathbb{R} \)
sequences \( x_n \) argument \( n \) is an integer, \( n \in \mathbb{Z} \)
ordered sequence \( (x_n)_n \)
set containing \( x_n \) \( \{x_n\}_n \)
vector \( x \) with \( x_n \) as elements \([x_n]\)
Dirac delta “function” \( \delta(t) \) \( \int x(t)\delta(t)\,dt = x(0) \)
Kronecker/Dirac/discrete impulse sequence \( \delta_n \) \( \delta_n = 1 \) for \( n = 0 \); \( \delta_n = 0 \) otherwise

**Elements of Real Analysis (TBD)**
integration by parts
\[
\int u\,dv = uv - \int v\,du
\]

**Elements of Complex Analysis (TBD)**
complex number \( z \) \( a + jb, re^{j\theta}, a, b \in \mathbb{R}, r \in \mathbb{R}^+, \theta \in [0, 2\pi] \)
conjugation \( z^* \) \( a - jb, re^{-j\theta} \)

\( X_n(z) \) conjugation of coefficients but not of \( z \)
\( W_N = e^{-j\frac{2\pi}{N}} \)

**Standard Vector Spaces**
Banach space of sequences with finite \( p \) norm, \( 1 \leq p < \infty \)
\( \ell^p(\mathbb{Z}) \) \( \{x: \mathbb{Z} \to \mathbb{C} \mid \sum_n |x_n|^p < \infty\} \) with norm \( \|x\|_p = (\sum_n |x_n|^p)^{1/p} \)
Banach space of bounded sequences with supremum norm
\( \ell^\infty(\mathbb{Z}) \) \( \{x: \mathbb{Z} \to \mathbb{C} \mid \sup_n |x_n| < \infty\} \) with norm \( \|x\|_\infty = \sup_n |x_n| \)
Banach space of functions with finite \( p \) norm, \( 1 \leq p < \infty \)
\( \mathcal{L}^p(\mathbb{R}) \) \( \{x: \mathbb{R} \to \mathbb{C} \mid \int |x(t)|^p\,dt < \infty\} \) with norm \( \|x\|_p = (\int |x(t)|^p\,dt)^{1/p} \)
Hilbert space of square-summable sequences
\( \ell^2(\mathbb{Z}) \) \( \{x: \mathbb{Z} \to \mathbb{C} \mid \sum_n |x_n|^2 < \infty\} \) with inner product \( \langle x, y \rangle = \sum_n x_ny_n^* \)
Hilbert space of square-integrable functions
\( \mathcal{L}^2(\mathbb{R}) \) \( \{x: \mathbb{R} \to \mathbb{C} \mid \int |x(t)|^2\,dt < \infty\} \) with inner product \( \langle x, y \rangle = \int x(t)y(t)^*\,dt \)

\(^0\) Last edit: JK Mar 02, 08
Bases and Frames

standard Euclidean basis \( \{ e_n \} \)

\( e_n = 1 \), for \( k = n \), and 0 otherwise

vector, element of basis or frame \( \varphi \)

when applicable, a column vector

basis or frame \( \Phi \)

set of vectors \( \{ \varphi_n \} \)

operator \( \Phi \)

concatenation of \( \varphi_n \)s in a linear operator: \([\varphi_0 \varphi_1 \ldots \varphi_{N-1}]\)

vector, element of dual basis or frame \( \varphi^* \)

when applicable, a column vector

operator \( \Phi^* \)

concatenation of \( \varphi_n^* \)s in a linear operator: \([\varphi_0^* \varphi_1^* \ldots \varphi_{N-1}^*]\)

expansion in a basis or frame \( x = \Phi\Phi^* x \)

Transforms

DFT: discrete Fourier trans.

\( x_n \overset{DFT}{\longleftrightarrow} X_k \)

\( X_k = \sum_{n=0}^{N-1} x_n W_N^{kn} \)

DTFT: discrete-time Fourier trans.

\( x_n \overset{DTFT}{\longleftrightarrow} X(e^{j\omega}) \)

\( X(e^{j\omega}) = \sum_{n=-\infty}^{\infty} x_n e^{-j\omega n} \)

FS: Fourier series

\( x(t) \overset{FS}{\longleftrightarrow} X_k \)

\( X_k = \frac{1}{T} \int_{-T/2}^{T/2} x(t)e^{-j2\pi kT} dt \)

FT: continuous-time Fourier trans.

\( x(t) \overset{CTFT}{\longleftrightarrow} X(\omega) \)

\( X(\omega) = \int_{-\infty}^{\infty} x(t)e^{-j\omega t} dt \)

ZT: z-trans.

\( x_n \overset{ZT}{\longleftrightarrow} X(z) \)

\( X(z) = \sum_{n=-\infty}^{\infty} x_n z^{-n} \)

Discrete-Time Nomenclature

sequence \( x_n \)

signal, vector

discrete-time system \( T_n \)

filter, operator

linear \( T_n \)

filter, operator, matrix

convolution \( h * x \)

convolution, filter, operator, matrix

eigensequence \( v_n \)

eigenfunction, eigenvector

finite time \( v_n = e^{j\omega n} \)

\( h * v = H(e^{j\omega}) v \)

finite time \( v_n = e^{j2\pi kn/N} \)

\( h * v = H_k v \)

frequency response \( H(e^{j\omega}) \)

infinite time \( \sum_{n=-\infty}^{\infty} h_n e^{-j\omega n} \)

finite time \( \sum_{n=0}^{N-1} h_n e^{-j\omega n} = \sum_{n=0}^{N-1} h_n W_N^{kn} \)

Filters

synthesis lowpass \( g_n \)

synthesis highpass \( h_n \)

analysis lowpass \( \tilde{g}_n \)

analysis highpass \( \tilde{h}_n \)

Two-Channel Filter Banks

lowpass sequence \( \alpha_k \)

\( \alpha_k = (\tilde{g}_{2k-n}, x_n) \)

highpass sequence \( \beta_k \)

\( \beta_k = (\tilde{h}_{2k-n}, x_n) \)

synthesis basis: even elements \( \varphi_{2k,n} = g_{n-2k} \)

synthesis basis: odd elements \( \varphi_{2k+1,n} = h_{n-2k} \)

analysis basis: even elements \( \varphi_{2k,n} = \tilde{g}_{n-2k} \)

analysis basis: odd elements \( \varphi_{2k+1,n} = \tilde{h}_{n-2k} \)

synthesis filter length \( L \)
Preface

The aim of these notes is to present, in a comprehensive way, a number of results, techniques, and algorithms for signal representation that have had a deep impact on the theory and practice of signal processing and communications. While rooted in classic Fourier techniques for signal representation, many results appeared during the flurry of activity of the 1980’s and 1990’s, when new constructions were found for local Fourier transforms and for wavelet orthonormal bases. These constructions were motivated both by theoretical interest and by applications, in particular in multimedia communications. New bases with specified time-frequency behavior were found, with impact well beyond the original fields of application. Areas as diverse as computer graphics and numerical analysis embraced some of the new constructions, no surprise given the pervasive role of Fourier analysis in science and engineering.

The presentation consists of two main parts, corresponding to background material and the central theme of signal representations. A companion book on applications is in the works.

Part I, Tools of the Trade, reviews all the necessary mathematical material to make the notes self-contained. For many readers, this material might be well known, for others, it might be welcome. It is a refresher of the basic mathematics used in signal processing and communications, and it develops the point of view used throughout the book. Thus, in Chapter 1, From Euclid to Hilbert, the basic geometric intuition central to Hilbert spaces is reviewed, together with all the necessary tools underlying the construction of bases. Chapter 2, Sequences and Signal Processing, is a crash course on processing signals in discrete time or discrete space. in Chapter 3, Fourier’s World, the mathematics of Fourier transforms and Fourier series is reviewed. The final chapter in Part I, Chapter 4, Sampling, Interpolation, and Approximation, talks about the critical link between discrete and continuous domains as given by the sampling theorem. It also veers from the exact world to the approximate one.

Part II, Fourier and Wavelet Representations, is the heart of the book. It aims at presenting a consistent view of signal representations that include Fourier, local Fourier, and wavelet bases, as well as related constructions, frames, and continuous transforms. It starts in Chapter 5, Time, Frequency, Scale and Resolution, with time-frequency analysis and related concepts, showing the intuitions
central to the signal representations constructed in the sequel. Chapter 6, Filter Banks: Building Blocks of Time-Frequency Expansions, presents a thorough treatment of the most elementary block—the two-channel filter bank, a signal processing device that splits a signal into a coarse, lowpass approximation, and a highpass difference. This block is then used to derive the discrete wavelet transform in Chapter 7, Wavelet Series on Sequences. It is also used to construct wavelets for the real line in Chapter 8, Wavelet Series on Functions, where other wavelet constructions are also given, in particular those based on multiresolution analysis. We then return to a more Fourier-like view of signal representations in Chapter 9, Localized Fourier Series on Sequences and Functions based on modulated filter banks. Relaxing the condition of completeness inherent in bases to allow for overcompleteness leads to frames, studied in Chapter 10, Frames on Sequences. Chapter 11, Continuous Wavelet and Windowed Fourier Transforms, develops continuous time-frequency transforms, where time, frequency, or scale indices are now continuous (and thus “infinitely” overcomplete!). The final Chapter 12, Approximation, Estimation, and Compression ends with three classical tasks, making a step towards the real world and modeling of that world; no small task. Fourier and wavelet representations are natural models for at least some objects of interest, and are thus shown in action.

As can be seen from the outline, we try to present a synthetic view from basic mathematical principles to actual construction of bases, always with an eye on concrete applications. While the benefit is a self-contained presentation, the cost is a rather sizable manuscript. We provide a reading guide with numerous routes through the material. The level spans from elementary to advanced material, but in a gradual fashion and with indications of levels of difficulty. In particular, starred sections can be skipped without breaking the flow of the material.

The material grew out of teaching signal processing, wavelets and applications in various settings. Two of the authors (Martin Vetterli and Jelena Kovačević) authored a graduate textbook, Wavelets and Subband Coding, Prentice Hall, 1995, which they and others used to teach graduate courses at various US and European institutions. With a decade of experience, the maturing of the field, and the broader interest arising from and for these topics, the time was right for a text geared towards a broader-audience, one that could be used to span levels from undergraduate to graduate, as well as various areas of engineering and science. As a case in point, parts of the text are used at Carnegie Mellon University in an undergraduate class on bioimage informatics, where some of the students are biology majors. This plasticity of the text is one of the features which we aimed for, and that most probably differentiates the present book from many others. Another aim is to present side-by-side all methods which have arisen around signal representations, without favoring any in particular. The truth is that each representation is a tool in the toolbox of the practitioner, and the problem or application at hand ultimately decides which one is the best!
Reading Guide

Below we give suggestions on how to material could be covered in a standard, one-semester, course. Most of these scenarios have been taught already by one of the authors (where appropriate, we will note that). We will also note levels and audience whenever possible.

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⁰Last edit: JK Mar 04 08

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