Nonlinear Control Systems

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6. Nonlinear Design

IST-DEEC PhD Course

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Sliding mode control

Consider the following system

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= h(x) + g(x)u\end{aligned}$$

where h, g are unknown nonlinear functions and $g(x) \geq g_0 > 0, \ \forall x$.

Goal: Design a state-feedback control law to stabilize the origin.

Idea: Design a control law that restrict the motion of the system to the manifold or surface

$$s = a_1 x_1 + x_2 = 0, \quad a_1 > 0$$

Note that the motion on the manifold $s = 0$ satisfies

$$x_2 = -a_1 x_1 \quad \longrightarrow \quad \dot{x}_1 = -a x_1 \quad \longrightarrow \quad x = (x_1, x_2) \rightarrow 0$$

and furthermore the motion is independent of h and g!

Now the question is how can we bring the trajectory to the manifold?

Sliding mode control

Let

$$V = \frac{1}{2}s^2$$

and therefore

$$\begin{aligned}\dot{V} &= s\dot{s} \\ &= s(a_1\dot{x}_1 + \dot{x}_2) \\ &= s(a_1x_2 + h(x)) + sg(x)u\end{aligned}$$

Suppose that $\left|\frac{a_1x_2+h(x)}{g(x)}\right| \leq \rho(x),\ \forall x \in \mathbb{R}^2$ and assume that $\rho(x)$ is known. Then,

$$\begin{aligned}\dot{V} &\leq |s|\rho(x)g(x) + su \\ &= g(x)|s|[\rho(x) + \text{sgn}(s)u]\end{aligned}$$

Let $u = -\beta(x)\text{sgn}(s)$ with $\beta(x) \geq \rho(x) + \beta_0,\ \beta_0 > 0$.

$$\dot{V} \leq -g(x)|s|\beta_0 \leq -g_0\beta_0|s|$$

Sliding mode control

Let $W = \sqrt{2V} = |s|$ (Note that $\sqrt{u}' = \frac{u'}{2\sqrt{u}}$)

The upper right-hand derivative is given by

$$D^+W = \frac{2\dot{V}}{2\sqrt{2V}} = \frac{\dot{V}}{W} \leq -g_0\beta_0\frac{W}{W}$$

By the comparison lemma

$$W(s(t)) \leq W(s(0)) - g_0\beta_0 t$$

Thus, the trajectory reaches the manifold $s = 0$ in finite time.
Moreover, once it reaches the manifold $\dot{V} \leq -g_0\beta_0|s| = 0$, which means that it cannot leave from it.

Sliding mode control

In summary, for the example above, the sliding mode control strategy is composed by two phases:

1. reaching phase: the trajectory starting off the manifold $s = 0$ move toward it and reach it in finite time.
2. sliding phase: the motion is confined to the manifold $s = 0$ and the dynamics of the system are represented by the reduced-order model $\dot{x}_1 = -a_1 x_1$.

Remark: The control law $u = -\beta(x)\text{sgn}(s)$ is called a sliding mode control law. Note that it is robust with respect to uncertainty on h and g. We only need to know the upper form $\rho(x)$.
Furthermore, if $\left|\frac{a_1 x_2 + h(x)}{g(x)}\right| \leq k_1, \forall x \in D$ then $u = -k\text{sgn}(s),\ k > k_1$ and if k can be chosen arbitrarily large, it can achieve semi-global stability.

However, due to imperfections in switching devices and delays, sliding mode control suffers from chattering!

What are the consequences of this zig-zag motion (oscillation)?

- High heat losses.
- High wear of moving mechanical parts.
- It may excite unmodelled high frequency dynamics
- Degrades performance and may lead to instability

To eliminate chattering replace u by

$$u = -\beta(x)\text{sat}(s/\epsilon)$$

where

$$\text{sat}(y) = \begin{cases} y, & \text{if } |y| \leq 1 \\ \text{sgn}(y), & \text{otherwise} \end{cases}$$

In that case we have,

$$\dot{V} \leq -g_0\beta_0|s|,$$

while $|s| \geq \epsilon$, which means that it reaches in finite time the set $\{|s| \leq \epsilon\}$.

Inside the boundary layer $|s| = \epsilon$, we have

$$s = a_1 x_1 + x_2 = a_1 x_1 + \dot{x}_1 \quad \longrightarrow \quad \dot{x}_1 = -a_1 x_1 + s$$

Thus, let

$$V_1 = \frac{1}{2} x_1^2$$

Then,

$$\begin{aligned} \dot{V} &\leq -a_1 x_1^2 + x_1 s \\ &\leq -a_1 x_1^2 + |x_1| \epsilon \\ &= -(1-\theta) a_1 x_1^2 - \theta a_1 x_1^2 + |x_1| \epsilon \\ &\leq -(1-\theta) a_1 x_1^2, \quad \forall |x_1| \geq \frac{\epsilon}{\theta a_1}, \ 0 < \theta < 1 \end{aligned}$$

Thus the trajectory reaches the set

$$\Omega_\epsilon = \left\{ |x_1| \leq \frac{\epsilon}{a_1 \theta}, \ |s| \leq \epsilon \right\}$$

in finite time, and therefore we can also conclude that it is ultimately bounded.

Sliding mode control

Stabilization

Consider the system

$$\dot{x} = f(x) + B(x)[G(x)E(x)u + \delta(t, x, u)]$$

where $x \in \mathbb{R}^n$ is the state, $u \in \mathbb{R}^p$ is the input, f, B, G and E are sufficiently smooth functions, and G and δ are unknown (uncertainties). Consider also that G is diagonal and positive definite with $g_i(x) \geq g_0 > 0$, $E(x)$ is nonsingular, $f(0) = 0$, and $x = 0$ is on open-loop equilibrium point (with $\delta = 0$).

Suppose that there is a diffeomorphic coordinate transformation $\begin{bmatrix} \eta \\ \xi \end{bmatrix} = T(x)$ (that is, $\frac{\partial T}{\partial x}$ is nonsingular, and T is proper, i.e, $\lim_{\|x\| \to \infty} \|T(x)\| = \infty$), with $\eta \in \mathbb{R}^{n-p}$, $\xi \in \mathbb{R}^p$ such that

$$\frac{\partial T}{\partial x} B(x) = \begin{bmatrix} 0 \\ I \end{bmatrix}$$

Then

$$\begin{bmatrix} \dot{\eta} \\ \dot{\xi} \end{bmatrix} = \frac{\partial T}{\partial x} f(x) + \frac{\partial T}{\partial x} B(x)[G(x)E(x)u + \delta(t, x, u)]$$

and so we obtain the system written in the so-called regular form:

$$\dot{\eta} = f_a(\eta, \xi) \quad (1)$$

$$\dot{\xi} = f_b(\eta, \xi) + G(x)E(x)u + \delta(t, x, u) \quad (2)$$

Step 1

Design the sliding manifold $s = \xi - \phi(\eta) = 0$ to stabilize (1), that is, when the motion is restricted to the manifold the reduced-order model

$$\dot{\eta} = f_a(\eta, \phi(\eta))$$

has an asymptotically stable equilibrium point at the origin.

This is the same as to solve the stabilization problem for the system

$$\dot{\eta} = f_a(\eta, \xi)$$

with ξ viewed as the control input.

Consider also that $\phi(\eta)$ is designed such that the system $\dot{\eta} = f_a(\eta, \phi(\eta) + s)$ is local ISS when s is viewed as the input.

Step 2

Design the control u to bring s to the boundary layer $\{|s_i| \leq \epsilon, 1 \leq i \leq p\}$ in finite time and keep it there $\forall t \geq T \geq 0$.

$$\dot{s} = \dot{\xi} - \frac{\partial \phi}{\partial \eta}\dot{\eta} = f_b(\eta, \xi) + G(x)E(x)u + \delta(t, x, u) - \frac{\partial \phi}{\partial \eta} f_a(\eta, \xi)$$

Let

$$u = E^{-1}(x)\hat{G}^{-1}(x)[f_b(\eta, \xi) - \frac{\partial \phi}{\partial \eta} f_a(\eta, \xi)] + E^{-1}(x)v$$

Then,

$$\dot{s}_i = g_i(x)v_i + \Delta_i(t, x, v), \quad i = 1, ..., p$$

where $\Delta_i(t, x, v)$ is the ith component of
$\Delta(t, x, v) = [I - G(x)\hat{G}^{-1}(x)][f_b(.) - \frac{\partial \phi}{\partial \eta} f_a(.)] + \delta(t, x, u)$
Assume that

$$\left| \frac{\Delta_i(t, x, u)}{g_i(x)} \right| \leq \rho(x) + k_0 \|v\|_\infty$$

with $k_0 \in [0, 1]$.

Then

$$V_i = \frac{1}{2}s_i^2$$

$$\begin{aligned}\dot{V}_i &= s_i\dot{s}_i \\ &= s_i g_i(x) v_i + s_i \Delta_i(t,x,u) \\ &\le |s_i| g_i(x)[v_i \mathsf{sgn}(s_i) + \rho(x) + k_0\|v\|_\infty]\end{aligned}$$

Take $v_i = -\beta(x)\mathsf{sat}(\frac{s_i}{\epsilon})$

$$\dot{V}_i \le |s_i| g_i(x) \left[-\beta(x)\mathsf{sat}\Big(\frac{|s_i|}{\epsilon}\Big) + \rho(x) + k_0\beta(x)\right]$$

In the region $|s_i| \ge \epsilon$ we have

$$\begin{aligned}\dot{V}_i &\le |s_i| g_i(x)\left[-(1-k_0)\beta(x) + \rho(x)\right] \\ &\le -g_0\beta_0(1-k_0)|s_i|\end{aligned}$$

by setting $\beta(x) \ge \frac{\rho(x)}{1-k_0} + \beta_0$, $\beta_0 > 0$.

Thus, $|s_i(t)|$ will decrease until it reaches the set $\{|s_i| \le \epsilon\}$ in finite time and remains inside thereafter.
We can conclude that the sliding mode controller achieves ultimate boundedness with an ultimate bound that can be controlled by the design parameter ϵ. Moreover it is robust with respect to matched uncertainties.

Example:

$$\dot{x}_1 = x_2 + \theta_1 x_1 \sin x_2$$
$$\dot{x}_2 = \theta_2 x_2^2 + x_1 + u$$

where θ_1, θ_2 are unknown parameters and $|\theta_1| \le a$ and $|\theta_2| \le b$ with a, b known. Note that the system is already in the regular form.

Step 1

Design x_2 to robustly stabilize $x_1 = 0$

$$\dot{x}_1 = x_2 + \theta_1 x_1 \sin x_2$$

$$V = \frac{1}{2} x_1^2$$

$$\dot{V} = x_1 x_2 + \theta_1 x_1^2 \sin x_2$$

Let $x_2 = -k x_1$. Then

$$\begin{aligned} \dot{V} &= -k x_1^2 + \theta_1 x_1^2 \sin(-k x_1) \\ &\leq -k x_1^2 + \theta_1 x_1^2 \\ &\leq -(k - a) x_1^2 \end{aligned}$$

for $k > a$. Thus, the sliding manifold is

$$s = x_2 + k x_1 = 0$$

Step 2

$$V = \frac{1}{2}s^2$$

$$\dot{V} = s[\theta_2 x_2^2 + x_1 + u + k(x_2 + \theta_1 x_1 \sin x_1)]$$

Let

$$u = -x_1 - kx_2 + v$$

$$\dot{V} = s[\theta_2 x_2^2 + k\theta_1 x_1 \sin x_2 + v]$$

where $|\Delta(x)| = |\theta_2 x_2^2 + k\theta_1 x_1 \sin x_2| \leq ak|x_1| + bx_2^2$.

Choose $v = -\beta(x)\mathsf{sat}(\frac{s}{\epsilon})$ with $\beta(x) = ak|x_1| + bx_2^2 + \beta_0, \;\; \beta_0 > 0$.
Then for $|s| \geq \epsilon$

$$\begin{aligned}\dot{V} &\leq |s| \left(|\Delta(x)| - \beta(x)\mathsf{sat}(\frac{|s|}{\epsilon}) \right) \\ &\leq -\beta_0 |s|\end{aligned}$$

so $|s|$ reaches in finite time the boundary layer $\{|s| \leq \epsilon\}$.

Example: Tracking

Consider the SISO system

$$\begin{aligned} \dot{x} &= f(x) + g(x)[u + \delta(t, x, u)] \\ y &= h(x) \end{aligned}$$

Normal form

$$\begin{aligned} \dot{\eta} &= f_0(\eta, \xi) \\ \dot{\xi}_1 &= \xi_2 \\ \dot{\xi}_2 &= \xi_3 \\ &\vdots \\ \dot{\xi}_{\rho-1} &= \xi_\rho \\ \dot{\xi}_\rho &= a(x) + b(x)[u + \delta(t, x, u)] \\ y &= \xi_1 \end{aligned}$$

Suppose that $\dot{\eta} = f_0(\eta, \xi)$ is ISS with ξ as input.
Goal: Track the reference $r(t)$ and suppose that $\dot{r}$, $\ddot{r}, \ldots, r^{(\rho)}$ are available.

Define

$$
\begin{aligned}
e_1 &= \xi_1 - r \\
e_2 &= \xi_2 - \dot{r} \\
&\vdots \\
e_\rho &= \xi_\rho - r^{\rho-1}
\end{aligned}
$$

Then

$$
\begin{aligned}
\dot{\eta} &= f_0(\eta, \xi) && (3) \\
\dot{e}_1 &= e_2 && (4) \\
&\vdots \\
\dot{e}_{\rho-1} &= e_\rho && (5) \\
\dot{e}_\rho &= a(x) + b(x)[u + \delta(t, x, u)] - r^{(\rho)}(t) && (6)
\end{aligned}
$$

Note that (3) is ISS and (4)-(5) is as a linear system (written in the controllable canonical form) with e_ρ viewed as input.

Therefore, for the linear subsystem select the linear control law

$$e_\rho = -(k_1 e_1 + k_2 e_2 + ... + k_{\rho-1} e_{\rho-1})$$

where k_1 to $k_{\rho-1}$ are chosen such that the closed-loop system is Hurwitz. Then, the sliding manifold is

$$s = k_1 e_1 + ... + k_{\rho-1} e_{\rho-1} + e_\rho = 0$$

Note that for $\rho = 2$, we have $s = k_1 e_1 + e_2 = k_1 e_1 + \dot{e}_1 = 0$.

To converge to the sliding manifold, choose

$$V = \frac{1}{2} s^2$$

Then,

$$\dot{V} = s\left(k_1 e_1 + ... + k_{\rho-1} e_{\rho-1} + e_\rho + a(x) + b(x)[u + \delta(t, x, u)] - r^{(\rho)}\right)$$

Let

$$u = -\frac{1}{b(x)}\left(k_2 e_2 + ... + k_{\rho-1} e_\rho - r^{(\rho)} + v\right)$$

$$\dot{V} = s\left(v + b(x)\delta(t, x, -\frac{1}{b(x)}[k_1 e_2.....r^{(\rho)}] + v)\right)$$

Defining $\Delta(t, x, v) = b(x)\delta(t, x, -\frac{1}{b(x)}[k_1 e_2.....r^{(\rho)}] + v$
if $|\Delta(t, x, v)| \leq \rho(x) + k_0|v|$, with $k_0 \in [0, 1)$. Then, selecting

$$v = -\beta(x)\text{sat}(\frac{s}{\epsilon}),\ \beta(x) \geq \frac{\rho(x)}{1 - k_0} + \beta_0,\ \beta_0 > 0$$

we can conclude that there exists a finite time $T \geq t_0$ such that the tracking error $|y(t) - r(t)|$ will be trapped inside a small neighborhood (that depends on ϵ) for all $t \geq T$.

Sliding mode with integral control

If the reference signal $r(t) = r$ is constant, we can achieve zero steady-state error using integral control.

To this effect define $e_0(t) = \int_0^t \big(y(\tau) - r\big) d\tau \quad \longrightarrow \quad \dot{e}_0 = y - r.$

Then we have

$$
\begin{aligned}
\dot{\eta} &= f_0(\eta, \xi) \\
\dot{e}_0 &= e_1 \\
\dot{e}_1 &= e_2 \\
&\vdots \\
\dot{e}_{\rho-1} &= e_\rho \\
\dot{e}_\rho &= a(x) + b(x)[u + \delta(\cdot)]
\end{aligned}
$$

and

$$s = k_0 e_0 + k_1 e_1 + ... + k_{\rho-1} e_{\rho-1} + e_\rho$$

In fact, for $\beta(x) = k$ and $v = k\text{sat}(\frac{s}{\epsilon})$ we have that for $\rho = 1$(relative degree one) the control algorithm turns out to be a classical Proportional Integral (PI) + saturation feedback law.
If the relative degree is two ($\rho = 2$) then we obtain a PID (Proportional, Integral and Derivative) structure + saturation

Nonlinear Lyapunov based control

Example: Pose stabilization of a fully actuated Autonomous Underwater Vehicle (AUV)

Consider the model of a fully actuated AUV

$$\begin{aligned} M\,\dot{\nu} + C(\nu)\,\nu + D(\nu)\,\nu + g(\eta) &= \tau \\ \dot{\eta} &= J(\eta)\,\nu \end{aligned}$$

where $\tau \in \mathbb{R}^6$ is the control input (forces and torques), $\eta \in \mathbb{R}^6$ is the position and orientation, $\nu \in \mathbb{R}^6$ is the linear and angular velocities, $M = M^T > 0$ is the rigid body and added mass inertia matrix, $C(\nu) = -C(\nu)^T$ is the matrix of Coriolis and Centrifugal terms, $D(\nu) > 0$ is the damping matrix, and $g(\eta)$ is the restoring term (buoyancy and gravity).

Goal: Design a state feedback control so that $\eta(t)$ converges to a desired position and attitude η_d (Pose stabilization)

Model:

$$M\,\dot{\nu} + C(\nu)\,\nu + D(\nu)\,\nu + g(\eta) = \tau$$
$$\dot{\eta} = J(\eta)\,\nu$$

Error Dynamics:

$$e(t) = \eta(t) - \eta_d(t) \quad \longrightarrow \quad \dot{\eta} = \dot{\eta} = J(\eta)\nu$$

Control Lyapunov Function (CLF):

$$V(\nu, e) = \frac{1}{2}\,\left(\nu^T\,M\,\nu + e^T\,K_P\,e\right)$$

Computing the time derivative with respect to the trajectory of the system...

$$\begin{aligned}\dot{V} &= \nu^T\,M\,\dot{\nu} + \dot{e}^T\,K_P\,e \\ &= \nu^T\left[M\,\dot{\nu} + J^T(\eta)\,K_P\,e\right] \\ &= \nu^T\left[\tau - D(\nu)\,\nu - g(\eta) + J^T(\eta)\,K_P\,e\right] - \nu^T\,C(\nu)\,\nu\end{aligned}$$

Assign the feedback law...

$$\tau = -J^T K_P\, e(t) - K_D\, \nu + g(\eta)$$

and we obtain

$$\dot{V} = -\nu^T\, [D(\nu) + K_D]\, \nu \,\leq\, 0$$

Thus the origin $(e, \nu) = 0$ is stable.

Can we prove Asymptotic Stability ?

Use LaSalle's invariance principle...

$$E = \{(\nu, e) \in \mathbb{R}^{12} : \nu = 0\} \quad \longrightarrow \quad 0 = J^T(\eta)\, K_P\, e$$

The largest invariant set M in E is the origin, thus we have asymptotic stability!

Therefore,

$$\lim_{t \to \infty} \eta(t) = \eta_d$$

Lyapunov redesign

Consider the system

$$\dot{x} = f(t,x) + G(t,x)[u + \delta(t,x,u)]$$

where $x \in \mathbb{R}^n$, $u \in \mathbb{R}^p$ and $\delta(\cdot)$ is an unknown disturbance that may depend on time, state, and input.

Suppose that for the nominal system

$$\dot{x} = f(t,x) + G(t,x)u$$

we have suceeded to design a feedback control law

$$u = \psi(t,x)$$

such that the origin $x = 0$ of

$$\dot{x} = f(t,x) + G(t,x)\psi(t,x)$$

is GUAS.

Lyapunov redesign

Furthermore, suppose that we have a C^1 function $V(t,x)$ that satisfies

$$\alpha_1(\|x\|) \leq V(t,x) \leq \alpha_2(\|x\|)$$
$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}[f(t,x) + G(t,x)\psi(t,x)] \leq -\alpha_3(\|x\|)$$

where $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $\alpha_3 \in \mathcal{K}$. Assume that with $u = \psi(t,x) + v$ the disturbance term δ satisfies

$$\|\delta(t,x,\psi(t,x)+v)\| \leq \rho(t,x) + k_0\|v\|, \quad 0 \leq k_0 < 1$$

where ρ is a nonnegative continuous function that estimates the size of the disturbance. Note that this is the only information about δ that we need to know.

Goal: Design an additional feedback control for v such that the overall control $u = \psi(t,x) + v$ stabilizes the actual system. The design of v is called Lyapunov redesign.

Closed-loop system:

$$\begin{aligned}\dot{x} &= f(t,x) + G(t,x)\left(\psi(t,x) + v + \delta(t,x,\psi(t,x)+v)\right) \\ &= f(t,x) + G(t,x)\psi(t,x) + G(t,x)\left(v + \delta(t,x,\psi(t,x)+v)\right)\end{aligned}$$

Thus

$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}[f(\cdot) + G(\cdot)\psi(\cdot)] + \frac{\partial V}{\partial x}G(\cdot)[v + \delta(\cdot)] \\ &\le -\alpha_3(\|x\|) + w^T v + w^T \delta\end{aligned}$$

where $w = \frac{\partial V}{\partial x}G(\cdot)$.

Goal: Make $w^T v + w^T \delta \le 0$.

To this effect note that

$$\begin{aligned}w^T v + w^T \delta &\le w^T v + \|w\|\|\delta\| \\ &\le w^T v + \|w\|[\rho(t,x) + k_0\|v\|]\end{aligned}$$

$$w^T v + w^T \delta \leq w^T v + \|w\|[\rho(t,x) + k_0\|v\|]$$

Let

$$v = \begin{cases} -\eta(t,x)\frac{w}{\|w\|}, & \eta(t,x)\|w\| \geq \varepsilon \\ -\eta^2(t,x)\frac{w}{\varepsilon}, & \eta(t,x)\|w\| < \varepsilon \end{cases}$$

with $\eta(t,x) \geq 0$.

Then for $\eta(t,x)\|w\| \geq \varepsilon$ we have

$$\begin{aligned} w^T v + w^T \delta &\leq -\eta(\cdot)\frac{\|w\|^2}{\|w\|} + \rho(\cdot)\|w\| + k_0\eta(\cdot)\frac{\|w\|}{\|w\|}\|w\| \\ &= -\eta(\cdot)[1-k_0]\|w\| + \rho(\cdot)\|w\| \end{aligned}$$

Choosing $\eta(t,x) \geq \frac{\rho(t,x)}{1-k_0}$ we obtain

$$w^T v + w^T \delta \leq -\rho(\cdot)\|w\| + \rho(\cdot)\|w\| = 0$$

Thus

$$\dot{V} \leq -\alpha_3(\|x\|)$$

For $\eta(t,x)\|w\| < \epsilon$, we have

$$\begin{aligned} w^T v + v^T \delta &\leq -\eta^2 \frac{\|w\|^2}{\epsilon} + \|w\|\rho + \eta^2 k_0 \frac{\|w\|^2}{\epsilon} \\ &= -(1-k_0)\frac{\eta^2}{\epsilon}\|w\|^2 + \|w\|\rho \\ &\leq (1-k_0)[-\frac{\eta^2}{\epsilon}\|w\|^2 + \eta\|w\|] \\ &= \left(\frac{1-k_0}{\epsilon}\right)\left(\eta\|w\| - \frac{\epsilon}{2}\right)^2 + \frac{(1-k_0)}{\epsilon}\frac{\epsilon^2}{4} \end{aligned}$$

Thus,

$$\dot{V} \leq -\alpha_3(\|x\|) + \frac{\epsilon}{4}(1-k_0).$$

We can now conclude that the solution is GUUB (Globally Uniformly Ultimated Bounded). That is,

$$\exists_{T \geq t_0 \geq 0} : \ \|x(t)\| \leq \beta(\|x(t)\|, t-t_0), \ \forall t_0 \leq t < T$$

and for $t \geq T$, $\|x(t)\| \leq \alpha(\epsilon)$. Note that if $\epsilon \to 0$ then $x \to 0$. In particular, we can have GUAS if there is a ball $B_a = \{\|x\| \leq a\}$ such that

$$\begin{aligned} \alpha_3(\|x\|) &\geq \phi^2(x),\ \phi(x) > 0 \\ \eta(t,x) &\geq \eta_0 > 0 \\ \rho(t,x) &\leq \rho_1 \phi(x) \end{aligned}$$

Then

$$\begin{aligned} \dot{V} &\leq -\alpha_3(\|x\|) - \frac{\eta^2}{\epsilon}(1-k_0)\|w\|^2 + \rho\|w\| \\ &\leq -\frac{1}{2}\alpha_3(\|x\|) - \frac{1}{2}\phi^2(x) - \frac{\eta_0^2}{\epsilon}(1-k_0)\|w\|^2 + \rho_1\phi(x)\|w\| \\ &\leq -\frac{1}{2}\alpha_3(\|x\|) - \frac{1}{2}\begin{bmatrix} \phi(x) \\ \|w\| \end{bmatrix}^T \begin{bmatrix} 1 & -\rho_1 \\ -\rho_1 & 2\eta_0^2\frac{(1-k_0)}{\varepsilon} \end{bmatrix} \begin{bmatrix} \phi(x) \\ \|w\| \end{bmatrix} \end{aligned}$$

In the case that

$$\epsilon < \frac{2\eta_0^2(1-k_0)}{\rho_1^2}$$

we have

$$\dot{V} \leq -\frac{1}{2}\alpha_3(\|x\|) < 0$$

and therefore $x = 0$ is GUAS.

Nonlinear Damping

Consider now the same system

$$\dot{x} = f(t,x) + G(t,x)[u + \delta(t,x,u)]$$

but with $\delta(t,x,u) = \Gamma(t,x)\delta_0(t,x,u)$, that is,

$$\dot{x} = f(t,x) + G(t,x)[u + \Gamma(t,x)\delta_0(t,x,u)]$$

where $\Gamma(\cdot)$ is known and δ_0 is bounded by $\|\delta_0(\cdot)\| \leq k_0$.

Let

$$u = \psi(t,x) + v$$

then

$$\begin{aligned}\dot{V} &= \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x}[f(\cdot) + G(\cdot)\psi(\cdot)] + \frac{\partial V}{\partial x}G(\cdot)[v + \Gamma(\cdot)\delta_0(\cdot)] \\ &\leq -\alpha_3(\|x\|) + w^T(v + \Gamma\delta_0)\end{aligned}$$

where $\alpha_3 \in \mathcal{K}_\infty$.

$$\dot{V} \le -\alpha_3(\|x\|) + w^T(v + \Gamma\delta_0)$$

Set

$$v = -kw\|\Gamma(t,x)\|^2, \quad k > 0 \quad \longleftarrow \quad \text{nonlinear damping}$$

which yields

$$\begin{aligned}
\dot{V} &\le -\alpha_3(\|x\|) - k\|w\|^2\|\Gamma(\cdot)\|^2 + \|w\|\|\Gamma(\cdot)\|k_0 \\
&= -\alpha_3(\|x\|) + \frac{k_0^2}{4k} \\
&= (1-\theta)\alpha_3(\|x\|) - \theta\alpha_3(\|x\|) + \frac{k_0^2}{4k}, \quad 0 < \theta < 1 \\
&\le -(1-\theta)\alpha_3(\|x\|), \quad \forall \|x\| \ge \alpha_3^{-1}(\frac{k_0^2}{4k\theta})
\end{aligned}$$

Thus it follows that the solutions are global uniformly ultimated bounded (GUUB).

Example

$$\dot{x} = x^2 + u + x\delta_0(t)$$

$$V = \frac{1}{2}x^2$$

$$\dot{V} = x[x^2 + u + x\delta_0(t)].$$

Set

$$u = -x^2 - x + v$$

then

$$\dot{V} = -x^2 + x[v + x\delta_0].$$

Set ($v = -kw\|\Gamma\|^2$)

$$v = -xx^2 = -x^3$$

then

$$\dot{V} = -x^2 - x^4 - x^2\delta_0 = -x^2 - (x^2 - \frac{1}{2}\delta_0)^2 + \frac{\delta_0^2}{4} \leq -x^2 + \frac{\delta_0^2}{4}$$

Note that the closed-loop system

$$\dot{x} = -x - x^3 + x\delta_0(t)$$

has a bounded solution no matter how large δ_0 is, and this thanks to the nonlinear damping $-x^3$.

Backstepping

Consider the system

$$\dot{\eta} = f(\eta) + g(\eta)\xi$$
$$\dot{\xi} = u$$

where $\eta \in \mathbb{R}^m$ and $\xi \in \mathbb{R}$ is viewed as a virtual input.

Suppose the first subsystem can be stabilized by a smooth state feedback law

$$\xi = \phi(\eta)$$

with $\phi(0) = 0$ that is, the origin of

$$\dot{\eta} = f(\eta) + g(\eta)\phi(\eta)$$

is GAS. Suppose further that we know a C^1 Lyapunov function $V(\eta)$ that satisfies

$$\frac{\partial V}{\partial \eta}[f(\eta) + g(\eta)\phi(\eta)] \leq -W(\eta)$$

Adding and subtracting $g(\eta)\phi(\eta)$,

$$\dot{\eta} = [f(\eta) + g(\eta)\phi(\eta)] + g(\eta)[\xi - \phi(\eta)]$$

Consider the change of variables

$$z = \xi - \phi(\eta)$$

then

$$\begin{aligned} \dot{\eta} &= [f(\eta) + g(\eta)\phi(\eta)] + g(\eta)z \\ \dot{z} &= u - \dot{\phi} \\ \dot{\phi} &= \frac{\partial \phi}{\partial \eta}[f(\eta) + g(\eta)\xi] \end{aligned}$$

Defining

$$v = u - \dot{\phi}$$

yields

$$\begin{aligned} \dot{\eta} &= [f(\eta) + g(\eta)\phi(\eta)] + g(\eta)z \\ \dot{z} &= v \end{aligned}$$

similar to the first system but now the origin of the first subsystem is GAS.

Consider the composite Lyapunov function

$$V_c = V(\eta) + \frac{1}{2}z^2$$

$$\begin{aligned}\dot{V}_c &= \frac{\partial V}{\partial \eta}[f(\eta) + g(\eta)\phi(\eta)] + \frac{\partial V}{\partial \eta}g(\eta)z + zv \\ &\leq -W(\eta) + [\frac{\partial V}{\partial \eta}g(\eta) + v]z\end{aligned}$$

Choosing

$$v = -\frac{\partial V}{\partial \eta}g(\eta) - kz, \quad k > 0$$

yields

$$\dot{V}_c \leq -W(\eta) - kz^2 < 0$$

Since $\xi = z + \phi(\eta)$ and $\phi(0) = 0$ then $(\eta = 0, z = 0)$ is GAS with

$$\begin{aligned}u &= v + \dot{\phi} \\ &= -\frac{\partial V}{\partial \eta}g(\eta) - k[\xi - \phi(\eta)] + \frac{\partial \phi}{\partial \eta}[f(\eta) + g(\eta)\xi]\end{aligned}$$

Example

$$\dot{x}_1 = x_1^2 - x_1^3 + x_2$$
$$\dot{x}_2 = u$$

Consider

$$V_1 = \frac{1}{2}x_1^2$$

$$\begin{aligned}\dot{V}_1 &= x_1(x_1^2 - x_1^3 + x_2) \\ &= -x_1^4 + x_1(x_1^2 + x_2)\end{aligned}$$

Remark: do not cancel $-x_1^4$ since it provides nonlinear damping.
Choose

$$x_2 = \phi(x_1) = -x_1^2 - x_1$$

then

$$\dot{V}_1 = -x_1^4 - x_1^2 \leq -x_1^2$$

which implies that $x_1 = 0$ is GES.

$$z_2 = x_2 - \phi(x_1) = x_2 + x_1^2 + x_1$$

Consider

$$V_2 = V_1 + \frac{1}{2}z_2^2 = \frac{1}{2}x_1^2 + \frac{1}{2}z_2^2$$

$$\begin{aligned}\dot{V}_2 &= x_1(-x_1^3 - x_1 + z_2) + z_2\big(u + (2x_1+1)(-x_1^3 - x_1 + z_2)\big) \\ &= -x_1^4 - x_1^2 + z_2\big(u + (2x_1+1)(-x_1^3 - x_1 + z_2) + x_1\big)\end{aligned}$$

Taking

$$u = -(2x_1+1)(-x_1^3 - x_1 + z_2) - x_1 - z_2$$

yields

$$\dot{V}_2 = -x_1^4 - x_1^2 - z_2^2$$

which implies that $x = 0$ is GAS.

Example: Third-order system

$$\dot{x}_1 = x_1^2 - x_1^3 + x_2$$
$$\dot{x}_2 = x_3$$
$$\dot{x}_3 = u$$

$$V_2 = \frac{1}{2}x_1^2 + \frac{1}{2}z_1^2$$

$$\dot{V}_2 = -x_1^2 - x_1^4 - z_2^2 + z_2 z_3$$

Let

$$V_3 = V_2 + \frac{1}{2}z_3^2$$

$$\dot{V}_3 = -x_1^2 - x_1^4 - z_2^2 + z_3[u - \dot{\phi} + z_2]$$

set

$$u = \dot{\phi} - z_2 - z_3$$

$$\dot{V}_3 = -x_1^2 - x_1^4 - z_2^2 - z_3^2 < 0$$

then $x = 0$ is GAS.

Recursive application of the backstepping

Strict-feedback form

$$\begin{aligned}
\dot{x}_1 &= x_2 + f_1(x_1) \\
\dot{x}_2 &= x_3 + f_2(x_1, x_2) \\
&\vdots \\
\dot{x}_i &= x_{i+1} + f_i(x_1, x_2, ..., x_i) \\
&\vdots \\
\dot{x}_n &= f_n(x_1, x_2, ..., x_n) + u
\end{aligned}$$

Idea: Consider the state x_2 as a virtual control input for x_1 and if it was a real input set

$$x_2 = -x_1 - f_1(x_1)$$

$$V_1 = \frac{1}{2}x_1^2 \quad \longrightarrow \quad \dot{V}_1 = -x_1^2$$

Define

$$
\begin{aligned}
z_1 &= x_1 \\
z_2 &= x_2 - \alpha(x_1), \quad \alpha(x_1) = -x_1 - f_1(x_1)
\end{aligned}
$$

Then

$$
\begin{aligned}
\dot{z}_1 &= -z_1 + z_2 \\
\dot{z}_2 &= x_3 + f_2(x_1, x_2) - \frac{\partial \alpha_1}{\partial x_1}(x_2 + f_1(x_1)) = x_3 + \bar{f}_2(z_1, z_2)
\end{aligned}
$$

$$
V_1 = \frac{1}{2} z_1^2
$$

$$
\dot{V}_1 = -z_1^2 + z_1 z_2
$$

Next step:

$$
\begin{aligned}
z_3 &= x_3 - \alpha_2(z_1, z_2) \\
V_2 &= V_1 + \frac{1}{2} z_2^2 \\
\dot{V}_2 &= -z_1^2 + z_2\big(z_1 + z_3 + \alpha_2(z_1, z_2) + \bar{f}_2(z_1, z_2)\big)
\end{aligned}
$$

$$\dot{V}_2 = -z_1^2 + z_2\big(z_1 + z_3 + \alpha_2(z_1, z_2) + \bar{f}_2(z_1, z_2)\big)$$

Choose $\alpha_2(z_1, z_2) = -z_1 - z_2 - \bar{f}_2(z_1, z_2)$ yields

$$\begin{aligned} \dot{z}_1 &= -z_1 + z_2 \\ \dot{z}_2 &= -z_1 - z_2 + z_3 \\ \dot{V}_2 &= -z_1^2 - z_2^2 + z_2 z_3 \end{aligned}$$

ith step:

$$\begin{aligned} z_{i+1} &= x_{i+1} - \alpha_i(z_1, ..., z_i) \\ V_i &= \frac{1}{2}(z_1^2 + z_2^2 + ... + z_i^2) \end{aligned}$$

$$\begin{aligned} \dot{z}_i &= z_i + \alpha_i(z_1, ..., z_i) + \bar{f}_i(z_1, ..., z_i) \\ V_i &= -z_1^2 - z_2^2 - ... - z_i^2 + z_{i-1} z_i + z_i\big(z_{i+1} + \alpha_i(z_1, ..., z_i) + \bar{f}_i(z_1, ..., z_i)\big) \end{aligned}$$

Using $\alpha(\cdot) = -z_{i-1} - z_i - \bar{f}_i(\cdot)$ it follows that

$$\dot{z}_i = -z_{i-1} - z_i + z_{i+1}$$
$$\dot{V}_i = -z_1^2 - ... - z_i^2 + z_i z_{i+1}$$

Last step:

$$\dot{z}_n = \bar{f}_n(z_1, ..., z_n) + u$$

Choose

$$u = \alpha_n(\cdot) = -z_{n-1} - z_n - \bar{f}_n(\cdot)$$

$$V_n = \frac{1}{2}(z_1^2 + \cdots + z_n^2)$$

$$\dot{z}_n = -z_{n-1} - z_n$$
$$\dot{V}_n = -z_1^2 - z_2^2 - \cdots - z_n^2$$

Example

$$\dot{\eta} = \eta^2 - \eta\xi$$
$$\dot{\xi} = u$$

$$V_1 = \frac{1}{2}\eta^2 \quad \longrightarrow \quad \dot{V}_1 = \eta(\eta^2 - \eta\xi)$$

if $\phi = \eta - \eta^2$, $z_2 = \xi - \phi$

$$\dot{V}_1 = -\eta^4 - \eta^2 z_2$$

$$V_2 = \frac{1}{2}\eta^2 + \frac{1}{2}z_2^2$$

$$\dot{V}_2 = -\eta^4 - \eta^2 z_2 + z_2[u - (\eta^2 - \eta\xi) - 2\eta(\eta^2 - \eta\xi)]$$

Choosing $u = (1 + 2\eta)(\eta^2 - \eta\xi) - kz_2 + \eta^2$ we obtain

$$\dot{V}_2 = -\eta^4 - kz_2 < 0$$

Block Backstepping

Consider the system

$$\dot{\eta} = f(\eta) + G(\eta)\xi$$
$$\dot{\xi} = f_a(\eta, \xi) + G_a(\eta, \xi)u$$

with $\eta \in \mathbb{R}^n$, $\xi \in \mathbb{R}^m$, $u \in \mathbb{R}^m$ and $G_a(\cdot)$ nonsingular.

Suppose that there exist $\phi(\eta)$, $\phi(0) = 0$ and $V(\eta)$ that satisfies

$$\frac{\partial V}{\partial \eta}[f(\eta) - G(\eta)\phi(\xi)] \leq -W(\eta)$$

Using

$$V_c = V(\eta) + \frac{1}{2}[\xi - \phi(\eta)]^T[\xi - \phi(\eta)]$$

$$\dot{V}_c = \frac{\partial V}{\partial \eta}(f + G\phi) + \frac{\partial V}{\partial \eta}G(\xi - \phi) + (\xi - \phi)^T[f_a + G_a u - \frac{\partial \phi}{\partial \eta}(f + G\xi)]$$

Taking

$$u = G_a^{-1}[\frac{\partial \phi}{\partial \eta}(f + G\xi) - (\frac{\partial V}{\partial \eta}G)^T - f_a - k(\xi - \phi)],\ k > 0$$

results in

$$\dot{V}_c \leq -W(\eta) - k[\xi - \phi(\eta)]^T[\xi - \phi(\eta)] < 0$$

and so $[\eta, \xi]^T = 0$ is GAS.