Nonlinear Control Systems

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5. Input-Output Stability

DEEC PhD Course

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2012

Input-Output Stability

$$y = Hu$$

- H denotes a mapping or operator that specifies y in terms of u
- u is an input signal that map the time interval $[0, \infty)$ into the Euclidean space $\mathbb{R}^m$, that is, $u : [0, \infty) \to \mathbb{R}^m$

Typical spaces of signals:

- $\mathcal{L}_\infty^m$ – space of piecewise continuous, bounded functions, where the norm of $u : [0, \infty) \to \mathbb{R}^m$ is defined as

$$\|u\|_{\mathcal{L}_\infty} = \sup_{t \geq 0} \|u(t)\| < \infty$$

- $\mathcal{L}_2^m$ – space of piecewise continuous, square-integrable functions with

$$\|u\|_{\mathcal{L}_2} = \sqrt{\int_0^\infty u^T(t) u(t) dt} < \infty$$

- $\mathcal{L}_p^m$ for $1 \leq p < \infty$ is the set of all piecewise continuous functions $u : [0, \infty) \to \mathbb{R}^m$ such that

$$\|u\|_{\mathcal{L}_p} = \left(\int_0^\infty \|u(t)\|^p dt \right)^{\frac{1}{p}} < \infty$$

Input-Output Stability

For technical reasons, we have to introduce the extended space

$$\mathcal{L}_e^m = \{u : u_\tau \in \mathcal{L}^m,\ \forall \tau \in [0, \infty)\}$$

where

$$u_\tau(t) = \begin{cases} u(t), & 0 \le t \le \tau \\ 0, & t > \tau \end{cases}$$

Note that u_τ is a truncation of u.

Example $u(t) = t \notin \mathcal{L}_\infty$ but its truncation belongs to $\mathcal{L}_\infty$ for every finite τ. Hence $u(t) = t \in \mathcal{L}_{\infty_e}$

Input-Output Stability

Definition

A mapping $H : \mathcal{L}_e^m \to \mathcal{L}_e^p$ is $\mathcal{L}$ stable if

$$\exists \alpha \in \mathcal{K}, \beta \geq 0 : \|(Hu)_\tau\|_\mathcal{L} \leq \alpha(\|u_\tau\|_\mathcal{L}) + \beta$$

for all $u \in \mathcal{L}_e^m$ and $\tau \in [0, \infty)$.

It is finite-gain $\mathcal{L}$ stable if

$$\exists \gamma, \beta \geq 0 : \|(Hu)_\tau\|_\mathcal{L} \leq \gamma \|u_\tau\|_\mathcal{L} + \beta$$

for all $u \in \mathcal{L}_e^m$ and $\tau \in [0, \infty)$.

In particular for $\mathcal{L}_\infty$, the system is $\mathcal{L}_\infty$ stable if for every bounded input $u(t)$, the output $Hu(t)$ is bounded. Given the system

$$\begin{aligned} &\dot{x} = f(t, x, u), \ x(0) = x_0 \\ &y = h(t, x, u) \end{aligned}$$

when is it $\mathcal{L}_\infty$ stable?

Theorem 5.3

Theorem 5.3

Consider the system

$$\dot{x} = f(t, x, u), \quad x(0) = x_0 \tag{1}$$

$$y = h(t, x, u) \tag{2}$$

Suppose that

- (1) is ISS, that is

$$\|x(t)\| \leq \beta(\|x_0\|, t - t_0) + \gamma\left(\sup_{\tau \in [0,t]} \|u(\tau)\|\right)$$

- h satisfies the inequality

$$\|h(t, x, u)\| \leq \alpha_1(\|x\|) + \alpha_2(\|x\|) + \eta$$

  $\alpha_1, \alpha_2 \in \mathcal{K}, \eta > 0$

Then, for each $x_0 \in \mathbb{R}^n$, system (1)-(2) is L_∞ stable.

Proof

$$\|y(t)\| \leq \alpha_1(\beta(\|x_0\|, t-t_0)) + \gamma(\sup_{\tau\in[0,t]} \|u(\tau)\|) + \alpha_2(\|u\|) + \eta$$

since $\alpha(a+b) \leq \alpha(2a) + \alpha(2b)$, then

$$\|y(t)\| \leq \alpha_1(2\beta(\|x_0\|, t-t_0)) + \alpha_1(2\gamma(\sup_{\tau\in[0,t]} \|u(\tau)\|)) + \alpha_2(\|u\|) + \eta$$

Thus

$$\|y_\tau\|_{L_\infty} \leq \gamma_0(\|u_\tau\|_{L_\infty}) + \beta_0$$

where

$$\gamma_0 = \alpha_1 \circ 2\gamma + \alpha_2, \;\; \beta_0 = \alpha_1(2\beta(\|x_0\|, 0)) + \eta$$

Example

Is the following system L_∞ stable?

$$\begin{aligned}\dot{x}_1 &= -x_1^3 + g(t)x_2 \\ \dot{x}_2 &= -g(t)x_1 - x_2^3 + u \\ y &= x_1 + x_2\end{aligned}$$

Consider

$$V = \frac{1}{2}x_1^2 + x_2^2$$

Thus,

$$\begin{aligned}\dot{V} &= -x_1^4 + g(t)x_1x_2 - g(t)x_1x_2 - x_2^4 + x_2u \\ &= -x_1^4 - x_2^4 + x_2u\end{aligned}$$

Since

$$\begin{aligned}\frac{1}{2}\|x\|^4 = \frac{1}{2}(x_1^2 + x_2^2)^{4/2} &= \frac{1}{2}x_1^4 + \frac{1}{2}x_2^4 + \frac{2}{2}x_1^2x_2^2 \\ &\leq \frac{1}{2}x_1^4 + \frac{1}{2}x_2^4 + \frac{1}{2}x_1^4 + \frac{1}{2}x_2^4 \\ &\leq x_1^4 + x_2^4\end{aligned}$$

Example

Then

$$\begin{aligned}\dot{V} &\le -\frac{1}{2}\|x\|^4 + \|x\||u| \\ &= -\frac{1}{2}(1-\theta)\|x\|^4 - \frac{1}{2}\theta\|x\|^4 + \|x\||u|,\ 0 < \theta < 1 \\ &\le -\frac{1}{2}(1-\theta)\|x\|^4,\ \forall \|x\| \ge \left(\frac{2|u|}{\theta}\right)^{1/3}\end{aligned}$$

Thus V is an ISS-Lyapunov function and the state equation is ISS.
Moreover

$$h(t) = \begin{bmatrix} 1 & 1 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} \to \|h(t)\| \le \sqrt{2}\|x\|$$

Thus the system is $\mathcal{L}_\infty$ stable.

L_2 Gain

Theorem 5.4

Consider the linear time-invariant system

$$\dot{x} = Ax + Bu$$
$$y = Cx + Du$$

where A Hurwitz. The L_2 gain of the linear system is given by

$$\gamma = \sup_{w \in \mathbb{R}} \|G(jw)\|_2 = \sqrt{\lambda_{\max}(G^T(-jw)G(jw))} = \sigma_{\max}(G(jw))$$

where $G(s) = C(sI - A)^{-1}B + D$. Moreover, in this case,

$$\|y\|_{L_2} = \gamma \|u\|_{L_2}$$

L_2 Gain

Theorem 5.5

Consider the nonlinear system

$$\begin{aligned} \dot{x} &= f(x) + G(x)u, \quad x(0) = x_0 \\ y &= h(x) \end{aligned}$$

where $f(x)$ is locally Lipschitz, $G \in \mathbb{R}^{n\times m}$ and $h : \mathbb{R}^n \to \mathbb{R}^q$ are continuous, and $f(0) = 0, h(0) = 0$.
Let γ be a positive number and suppose that there is a C^1, positive semidefinite function $V(x)$ that satisfies the Hamilton-Jacobi inequality

$$\frac{\partial V}{\partial x} f(x) + \frac{1}{2\gamma^2} G(x) G^T(x) \left(\frac{\partial V}{\partial x}\right)^T + \frac{1}{2} h^T(x) h(x) \leq 0$$

Then, $\forall x_0 \in \mathbb{R}^n$, the system is finite-gain L_2 stable and its L_2 gain is less than or equal to γ.

Proof

$$\dot{V}(x) = \frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} G(x) u$$
$$\leq -\frac{1}{2\gamma^2} \frac{\partial V}{\partial x} G(x) G^T(x) \left(\frac{\partial V}{\partial x}\right)^T - \frac{1}{2} h^T(x) h(x) + \frac{\partial V}{\partial x} G(x) u$$

By completing the squares, the right hand-side term is equal to

$$-\frac{1}{2}\gamma^2 \left\| u - \frac{1}{\gamma^2} G^T(x) \left(\frac{\partial V}{\partial x}\right)^T \right\|^2 + \frac{1}{2}\gamma^2 \|u\|^2 - \frac{1}{2}\|y\|^2$$

Hence

$$\frac{\partial V}{\partial x} f(x) + \frac{\partial V}{\partial x} G(x) u \leq \frac{1}{2}\gamma^2 \|u\|^2 - \frac{1}{2}\|y\|^2$$

Integrating

$$V(x(\tau)) - V(x_0) \leq \frac{1}{2}\gamma^2 \int_0^\tau \|u(\tau)\|^2 d\tau - \frac{1}{2} \int_0^\tau \|y(\tau)\|^2 d\tau$$

Since $V(x) \geq 0$

$$\frac{1}{2} \int_0^\tau \|y(\tau)\|^2 d\tau \leq \frac{1}{2}\gamma^2 \int_0^\tau \|u(\tau)\|^2 d\tau + 2V(x_0)$$

Taking the square roots and using $\sqrt{a^2 + b^2} \leq a + b$ for $a, b \geq 0$ yields

$$\|y_\tau\|_{L_2} \leq \gamma \|u_\tau\|_{L_2} + \sqrt{2V(x_0)}$$

The Small-Gain Theorem

Let $H_1 : L_e^m \to L_e^q$ and $H_2 : L_e^q \to L_e^m$ and

- Suppose both systems are finite-gain L stable, that is,

$$\|y_{1\tau}\|_L \leq \gamma_1 \|e_{1\tau}\|_L + \beta_1,\ \forall e_1 \in L_e^m,\ \forall \tau \in [0, \infty)$$
$$\|y_{2\tau}\|_L \leq \gamma_2 \|e_{2\tau}\|_L + \beta_2,\ \forall e_2 \in L_e^q,\ \forall \tau \in [0, \infty)$$

- Suppose that the feedback is well defined and let

$$u = \begin{bmatrix} u_1 \\ u_2 \end{bmatrix}, y = \begin{bmatrix} y_1 \\ y_2 \end{bmatrix}, e = \begin{bmatrix} e_1 \\ e_2 \end{bmatrix}$$

Then, the feedback connection is finite-gain L stable if $\gamma_1 \gamma_2 < 1$.

Proof

Let $e_{1\tau} = u_{1\tau} - (H_2 e_2)_\tau$ and $e_{2\tau} = u_{2\tau} - (H_1 e_1)_\tau$.
Then

$$\begin{aligned}
\|e_{1\tau}\|_L &\le \|u_{1\tau}\|_L + \|(H_2 e_2)_\tau\|_L \\
&\le \|u_{1\tau}\|_L + \gamma_2 \|e_{2\tau}\|_L + \beta_2 \\
&\le \|u_{1\tau}\|_L + \gamma_2 \left(\|u_{2\tau}\|_L + \gamma_1 \|e_{1\tau}\|_L + \beta_1\right) + \beta_2 \\
&= \gamma_1\gamma_2 \|e_{1\tau}\|_L + \left(\|u_{1\tau}\|_L + \gamma_2 \|u_{2\tau}\|_L + \beta_2 + \gamma_2\beta_1\right)
\end{aligned}$$

Thus

$$(1 - \gamma_1\gamma_2)\|e_{1\tau}\|_L = \|u_{1\tau}\|_L + \gamma_2 \|u_{2\tau}\|_L + \beta_2 + \gamma_2\beta_1$$

Since $\gamma_1\gamma_2 < 1$, then

$$\|e_{1\tau}\|_L \le \frac{1}{1 - \gamma_1\gamma_2} \left(\|u_{1\tau}\|_L + \gamma_2 \|u_{2\tau}\|_L + \beta_2 + \gamma_2\beta_1\right)$$

for all $\tau \in [0, \infty)$. Similarly,

$$\|e_{2\tau}\|_L \le \frac{1}{1 - \gamma_1\gamma_2} \left(\|u_{2\tau}\|_L + \gamma_1 \|u_{1\tau}\|_L + \beta_1 + \gamma_1\beta_2\right)$$

Thus, using

$$\|e\|_L \le \|e_1\|_L + \|e_2\|_L$$

we conclude the result.

Example 1

Suppose we have two systems:

- H1 : linear time-invariant system, where $G(s)$ is a Hurwitz square transfer function matrix
- H2: $y_2 = \phi(t, e_2)$ such that $\|\phi(t, y)\| \leq \gamma_2 \|y\|$

Some remarks

- H1 is finite gain L_2 stable with L_2 gain given by $\gamma_1 = \sup_{w \in \mathbb{R}} \|G(jw)\|_2$
- H2 is finite gain L_2 stable with L_2 gain less that or equal to γ_2

Thus, the interconnection (assuming that is well defined) is finite-gain L_2 stable if $\gamma_1 \gamma_2 < 1$

Example 2: Robustness of the controller with respect to unmodeled actuator dynamics

Consider the system

$$
\begin{aligned}
\dot{x} &= f(t, x, v + d_1(t)) && \longrightarrow && \text{plant dynamics} \\
\epsilon \dot{z} &= Az + B[u + d_2(t)] && \longrightarrow && \text{"fast" actuator dynamics} \\
v &= Cz
\end{aligned}
$$

where f is a smooth function, $\epsilon > 0$ is a small parameter (fast dynamics), A is Hurwitz, $-CA^{-1}B = I$, and $d_1, d_2 \in L_\infty$ are disturbance signals with $\dot{d}_1, \dot{d}_2 \in L_\infty$.

Goal: Attenuate the effect of the disturbance on the state x

This can be achieved if the feedback control law can be designed such that the closed-loop input-output map from (d_1, d_2) to x is finite-gain L stable with L gain less than some given tolerance $\delta > 0$.

Example 2

Neglecting the actuator dynamics by setting $\epsilon = 0$ we have $z = -A^{-1}B(u + d_2)$ and $v = -CA^{-1}B(u + d_2) = u + d_2$
Thus, we obtain

$$\dot{x} = f(t, x, u + d), \quad d = d_1 + d_2$$

Suppose we design a control law

$$u = K(t, x)$$

such that

$$\|x\|_{L_\infty} \leq \gamma \|d\|_{L_\infty} + \beta \tag{3}$$

for some $\gamma < \delta$.

Is the controller robust with respect to the unmodeled actuator dynamics?

Example 2

Closed-loop dynamics:

$$\begin{aligned} \dot{x} &= f(t, x, cz + d_1(t)) \\ \epsilon \dot{z} &= Az + B[K(t, x) + d_2(t)] \end{aligned}$$

Let η be the error, that is,

$$\eta = z - z_{|\epsilon=0} = z + A^{-1}B[K(t, x) + d_2(t)]$$

Then,

$$\begin{aligned} \epsilon \dot{\eta} &= Az + B[K(\cdot) + d_2] + \epsilon A^{-1}B[\dot{K}(\cdot) + \dot{d}_2] \\ &= A\eta - AA^{-1}B[K(\cdot) + d_2] + B[K(\cdot) + d_2] + \epsilon A^{-1}B[\dot{K}(\cdot) + \dot{d}_2] \end{aligned}$$

Let $e_2 := \dot{K}(\cdot) + \dot{d}_2$ then

$$\dot{\eta} = \frac{1}{\epsilon} A\eta + A^{-1}Be_2$$

Notice also that

$$\begin{aligned} \dot{x} &= f(t, x, C\eta - CA^{-1}B[K(\cdot) + d_2] + d_1) \\ &= f(t, x, K(\cdot) + C\eta + d) = f(t, x, K(\cdot) + e_1) \end{aligned}$$

where $-CA^{-1}B = I$ and $e_1 := C\eta + d$

Example 2

Assume that

$$\|\frac{\partial K}{\partial t} + \frac{\partial K}{\partial x} f(t, x, K(t, x) + e_1)\| \leq c_1\|x\| + c_2\|e_1\|$$

Then using (3)

$$\|y_1\|_{L_\infty} \leq c_1\gamma\|e_1\|_{L_\infty} + c_1\beta + c_2\|e_1\|_{L_\infty} = \gamma\|e_1\|_{L_\infty} + \beta_1$$

where $\gamma_1 = c_1\gamma + c_2$ and $\beta_1 = c_1\beta$. Since H2 is a linear system then

$$\|y_2\|_L \leq \epsilon\gamma_f\|e_2\|_{L_\infty} + \beta_2$$

Thus, we can conclude that if $\epsilon\gamma_1\gamma_f < 1$, then the closed-loop system is L_∞ stable.