Nonlinear Control Systems

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4. Lyapunov Stability

IST DEEC PhD Course

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Autonomous System

Consider the autonomous system

$$\dot{x} = f(x) \tag{1}$$

where $f : D \to \mathbb{R}^n$ is a locally Lipschitz map from a domain $D \subset \mathbb{R}^n$ into $\mathbb{R}^n$ and there is at least one equilibrium point $\bar{x}$, that is $f(\bar{x}) = 0$.

Goal: Stability analysis of the equilibrium point $\bar{x} \in D$.

Without loss of generality, we consider that $\bar{x} = 0$.
Why? If it is not, then consider the change of variables $y = x - \bar{x}$. Then

$$\dot{y} = f(x) = f(y + \bar{x}) := g(y)$$

where $g(0) = 0$

Definition

The equilibrium point $x = 0$ of (1) is

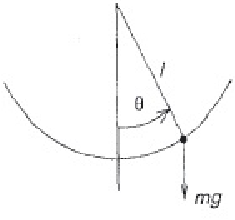
- stable if, for each $\epsilon > 0$, there is $\delta = \delta(\epsilon) > 0$ such that

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \quad \forall t \geq 0$$

- unstable if it is not stable
- asymptotically stable if it is stable and δ can be chosen such that

$$\|x(0)\| < \delta \Rightarrow \lim_{t \to \infty} x(t) = 0$$

Example - Pendulum



$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{l}\sin(x_1) - \frac{\beta}{m}x_2$$

where $x_1 = \theta,\ x_2 = \dot{\theta}$, and β is the friction. The following equilibrium points

- $x = (0,0)$ (which is a stable focus) is asymptotically stable
- $x = (\pi,0)$ (which is a saddle point) is unstable

For $\beta = 0$ then the above equilibrium points are stable (but not asymptotically).

A function $V : D \to \mathbb{R}$ is said to be

- positive definite if $V(0) = 0$ and $V(x) > 0,\ \forall x \neq 0$
- positive semidefinite if $V(0) = 0$ and $V(x) \geq 0,\ \forall x \neq 0$
- negative definite (resp. negative semi definite) if $-V(x)$ is definite positive (resp. definite semi positive).

In particular, for $V(x) = x^T P x$ (quadratic form), where P is a real symmetric matrix, $V(x)$ is positive (semi)definite if and only if all the eigenvalues of P are positive (nonnegative), which is true if all leading principal minors of P are positive (all principal minors of P are nonnegative).

Example

$$
\begin{aligned}
V(x) &= a x_1^2 + 2x_1 x_3 + a x_2^2 + 4 x_2 x_3 + a x_3^2 \\
&= \begin{bmatrix} x_1 & x_2 & x_3 \end{bmatrix} \begin{bmatrix} a & 0 & 1 \\ 0 & a & 2 \\ 1 & 2 & a \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \\ x_3 \end{bmatrix} = x^T P x
\end{aligned}
$$

The leading principal minors of P are a, $\begin{vmatrix} a & 0 \\ 0 & a \end{vmatrix} = a^2$, and

$$
\begin{vmatrix} a & 0 & 1 \\ 0 & a & 2 \\ 1 & 2 & a \end{vmatrix} = a^3 - a - 4a = a\left(a^2 - 5\right)
$$

Therefore, $V(x) > 0$ if $a > \sqrt{5}$.

Lyapunov's stability theorem

Theorem 4.1 - Lyapunov's stability theorem

Let $x = 0$ be an equilibrium point for $\dot{x} = f(x)$ and $D \subset \mathbb{R}^n$ be a domain containing $x = 0$. Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that

- $V(0) = 0$, $V(x) > 0, \quad \forall x \in D \setminus \{0\}$
- $\dot{V}(x) \leq 0, \quad \forall x \in D$

Then, $x = 0$ is stable. Moreover, if

$$\dot{V}(x) < 0, \forall x \in D \setminus \{0\}$$

then $x = 0$ is asymptotically stable.

Proof

- Given $\epsilon > 0$, chose $r \in (0, \epsilon]$ such that

$$B_r = \{x \in \mathbb{R}^n : \ \|x\| \leq r\} \subset D$$

- Let $\alpha = \min_{\|x\|=r} V(x)$. Then $\alpha > 0$.
  Take $\beta \in (0, \alpha)$ and let

$$\Omega_\beta = \{x \in B_r : V(x) \leq \beta\}$$

  Then $\Omega_\beta \subset B_r$.
  Why? Because if it was not, then there is a point $p \in \Omega_\beta$ that lies on the boundary of B_r. But $V(p) \geq \alpha > \beta$, although for all $x \in \Omega_\beta$, $V(x) \leq \beta$ which is a contradiction.

- The set Ω_β is an invariant set, that is, for any trajectory starting in Ω_β at $t = 0$ stays in Ω_β, $\forall t \geq 0$.
  Why? Because

$$\dot{V}(x(t)) \leq 0 \Rightarrow V(x(t)) \leq V(x(0)) \leq \beta, \ \forall t \geq 0$$

  Note also that there exists a unique solution defined for all $t \geq 0$ whenever $x(0) \in \Omega_\beta$ because Ω_β is a compact set (closed and bounded since it is contained in B_r).

Proof

- By continuity of $V(x)$ and $V(0) = 0$, we conclude that there is a $\delta > 0$ such that

$$\|x(t)\| \leq \delta \Rightarrow V(x) < \beta$$

Then,

$$B_\delta \subset \Omega_\beta \subset B_r$$

and

$$x(0) \in B_\delta \Rightarrow x(0) \in \Omega_\beta \Rightarrow x(t) \in \Omega_\beta \Rightarrow x(t) \in B_r$$

Therefore,

$$\|x(0)\| < \delta \Rightarrow \|x(t)\| < r \leq \epsilon, \quad \forall t \geq 0$$

which shows that the equilibrium point $x = 0$ is stable.

Proof

To prove asymptotically stability and assuming that $\dot{V}(x) < 0 \ \forall x \in D \backslash \{0\}$ we have to show that $x(t) \to 0$ as $t \to \infty$, that is

$$\forall a > 0 \ \exists \tau > 0 : \|x(t)\| < a, \quad \forall t > \tau$$

But

$$\forall a > 0 \ \exists b > 0 : \Omega_b \subset B_a$$

So it is sufficient to show that

$$V(x(t)) \to 0 \text{ as } t \to \infty$$

Since V is monotonically decreasing and bounded from below by zero, then

$$V(x(t)) \to c \geq 0 \text{ as } t \to \infty$$

We need to show that $c = 0$. By contradiction, suppose that $c > 0$, which implies that the trajectory lies outside the ball $B_d \subset \Omega_c$. But

$$V(x) = V(x(0)) + \int_0^t \dot{V}(x(\tau)) d\tau \leq V(x(0)) - \gamma t$$

because $\dot{V}(x(\tau)) \leq -\gamma = \max_{d \leq \|x\| \leq r} \dot{V}(x)$. Thus, V will eventually become negative, which is a contradiction since $V > 0$.

Examples

1.

$$\dot{x} = -g(x), \quad x \in \mathbb{R}$$

with g locally Lipschitz on $(-a, a)$ and

$$g(0) = 0, \quad xg(x) > 0, \quad \forall x \neq 0, x \in (-a, a)$$

Let

$$V(x) = \int_0^x g(y)dy$$

It is continuously differentiable and positive definite. Thus, $V(x)$ is a valid *Lyapunov function candidate*. To check if it is a Lyapunov function, we compute

$$\dot{V}(x) = -\frac{\partial V}{\partial x} g(x) = -g(x)^2 < 0, \quad \forall x \in D \backslash \{0\}$$

Hence, the origin is asymptotically stable.

Examples

2. Consider the Pendulum example without friction

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -a \sin(x_1) \end{aligned}$$

Assume the following energy function

$$V(x) = \int_0^{x_1} a \sin(y) dy + \frac{1}{2} x_2^2 = a(1 - \cos(x_1)) + \frac{1}{2} x_2^2.$$

Clearly, $V(0) = 0$ and $V(x) > 0$, $-2\pi < x_1 < 2\pi$, $x_1 \neq 0$.

$$\begin{aligned} \dot{V}(x) &= a \sin(x_1) \dot{x}_1 + x_2 \dot{x}_2 \\ &= a \sin(x_1) x_2 - x_2 a \sin(x_1) = 0 \end{aligned}$$

Thus the origin is stable. Since $\dot{V}(x) = 0$, we can also conclude that the origin is not asymptotically stable.

Examples

3. Pendulum equation, but this time with friction

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -a\sin(x_1) - bx_2$$

Consider

$$V(x) = a(1 - \cos(x_1)) + \frac{1}{2}x_2^2$$

Then,

$$\dot{V}(x) = -bx_2^2 \leq 0$$

which is negative semidefinite.
why? because $\dot{V}(x) = 0$ for $x_2 = 0$ irrespective of the value of x_1.

We can only conclude that the origin is stable!

Examples

3. However, we know that is asymptotically stable. Let us try

$$V(x) = \frac{1}{2}x^T P x + a(1 - \cos(x_1))$$

where P given by

$$P = \begin{bmatrix} P_{11} & P_{12} \\ P_{12} & P_{22} \end{bmatrix}$$

is positive definite if $P_{11} > 0$ and $P_{11}P_{22} - P_{12}^2 > 0$.
Computing $\dot{V}$ and taking $P_{22} = 1$, $P_{11} = bP_{12}$, $P_{12} = b/2$, yields

$$\dot{V} = -\frac{1}{2}a\,b\,x_1\sin(x_1) - \frac{1}{2}b\,x_2^2$$

The term $x_1\sin(x_1) > 0$ for all $0 < |x_1| < \pi$.
Taking $D = \{x \in \mathbb{R}^2 : |x_1| < \pi\}$ we conclude that V is a Lyapunov function and the origin is asymptotically stable.
This example emphasizes an important feature:

The Lyapunov theorem's conditions are only sufficient!

Region of attraction

Let $\phi(t, x)$ be the solution of $\dot{x} = f(x)$ that starts at initial state x at time $t = 0$. Then, the region of attraction is defined as the set of all points x such that $\phi(t, x)$ is defined for all $t \geq 0$ and

$$\lim_{t \to \infty} \phi(t, x) = 0.$$

If the Lyapunov function satisfies the conditions of asymptotic stability over a domain D, then the set

$$\Omega_c = \{x \in \mathbb{R}^n : V(x) \leq c\} \subset D$$

is an estimate of the region of attraction.

When the region of attraction is $\mathbb{R}^n$?
That is, when $x = 0$ is globally asymptotically stabe (GAS)?
Clearly $D = \mathbb{R}^n$, but is this enought?

NO!

$$V(x) = \frac{x_1^2}{1+x_1^2} + x_2^2$$

Ω_c is unbounded with c large! For Ω_c to be in the interior of a ball B_r, c must satisfies

$$c < \inf_{\|x\| \geq r} V(x).$$

If

$$l = \lim_{r\to\infty} \inf_{\|x\| \geq r} V(x) < \infty$$

then Ω_c is bounded only if $c < l$. In the example

$$l = \lim_{r\to\infty} \min_{\|x\|=r} [\frac{x_1^2}{1+x_1^2} + x_2^2] = \lim_{\|x_1\|\to\infty} [\frac{x_1^2}{1+x_1^2}] = 1$$

Thus Ω_c is bounded only for $c < 1$. To ensure that Ω_c is bounded *for all values* of $c > 0$ we need the radially unbounded condition

$$V(x) \to \infty \; as \; \|x\| \to \infty$$

Laypunov Stability - Globally Asymptotically Stability

Theorem 4.2 - GAS

Let $x = 0$ be an equilibrium point for $\dot{x} = f(x)$. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function such that

- $V(0) = 0$ and $V(x) > 0, \quad \forall x \neq 0$
- $\|x\| \to \infty \Rightarrow V(x) \to \infty$
- $\dot{V}(x) < 0, \quad \forall x \neq 0.$

Then $x = 0$ is globally asymptotically stable (GAS).

Instability Theorem

Theorem 4.3 - Instability Theorem

Let $x = 0$ be an equilibrium point for $\dot{x} = f(x)$. Let $V : D \to \mathbb{R}$ be a continuously differentiable function such that $V(0) = 0$ and $V(x_0) > 0$ for some x_0 with arbitrarily small $\|x_0\|$. Choose $r > 0$ such that the ball $B_r = \{x \in \mathbb{R}^n : \|x\| \leq r\}$ is contained in D. Define a set U given by

$$U = \{x \in B_r : V(x) > 0\}$$

and suppose that $\dot{V}(x) > 0$ in U. Then, $x = 0$ is unstable.

The Invariant Principle

Definitions

- P is said to be a *positive limit point* of $x(t)$ if there is a sequence $\{t_n\}$, with $t_n \to \infty$ as $x \to \infty$ such that $x(t_n) \to p$ as $n \to \infty$.
- The set of all positive limit points is called *positive limit set*.
- A set M is said to be a *positively invariant set* if

$$x(0) \in M \Rightarrow x(t) \in M, \quad \forall t \geq 0$$

- $x(t)$ approaches a set M as $t \to \infty$, if

$$\forall \epsilon > 0 \, \exists T > 0 : \text{dist}(x(t), M) < \epsilon, \quad \forall t > T$$

  where $\text{dist}(p, M) = \inf_{x \in M} \|p - x\|$.

Lemma 4.1

If a solution $x(t)$ of $\dot{x} = f(x)$ is bounded and belongs to D for $t \geq 0$, then its positive limit set L^* is a nonempty, compact, invariant set. Moreover, $x(t)$ approaches L^* as $t \to \infty$.

La Salle's Theorem

Theorem 4.11 - La Salle's Theorem

Let

- $\Omega \subset D$ be a compact positively invariant set.
- $V : D \to \mathbb{R}$ be a continuously differentiable function such that $\dot{V}(x) \leq 0$ in Ω.
- $E = \{x \in \Omega : \dot{V}(x) = 0\}$.
- M be the largest invariant set in E.

Then every solution starting in Ω approaches M as $t \to \infty$.

Proof.

(Outline)

- continuity of $V(x)$		- $V(x) \to a$
- Ω is bounded and closed	$\Longrightarrow$	- $L^+ \subset \Omega$
- $\dot{V}(x) = 0$		- $V(x) = a$ on L^+

From Lemma 4.1 we have that L^+ is an invariant set, $\dot{V}(x) = 0$ on L^+. Thus, we conclude that $L^* \subset M \subset E \subset \Omega$. Since $x(t)$ is bounded, $x(t) \to L^+$ as $t \to \infty$ (by Lemma 4.1). Hence, $x(t) \to M$ as $t \to \infty$. □

Note that $V(x)$ is not needed to be positive definite.

When E is the origin?

Corollary 4.1

Let $x = 0$ be an equilibrium point of $\dot{x} = f(x)$. Let $V : D \to \mathbb{R}$ be a C^1 positive definite function containing the origin $x = 0$ such that $\dot{V} \leq 0$ in D. Let $S = \{x \in D : \dot{V} = 0\}$ and suppose that no solution can stay identically in S, other than the trivial solution $x(t) = 0$. Then, the origin is asymptotically stable.

Corollary 4.2

Let $x = 0$ be an equilibrium point of $\dot{x} = f(x)$. Let $V : \mathbb{R}^n \to \mathbb{R}$ be a C^1, radially unbounded, positive definite function such that $\dot{V} \leq 0$ for all $x \in \mathbb{R}^n$. Let $S = \{x \in \mathbb{R}^n : \dot{V} = 0\}$ and suppose that no solution can stay identically in S, other than the trivial solution $x(t) = 0$. Then $x = 0$ is GAS.

Example

$$\begin{aligned}\dot{x}_1 &= x_2 \\ \dot{x}_2 &= -h_1(x_1) - h_2(x_2)\end{aligned}$$

where $h_i(0) = 0$ and $y\, h_i(y) > 0, \forall y \neq 0$, $i = 1, 2$. Also assume that $\int_0^y h_1(z)dz \to \infty$ as $\|y\| \to \infty$.

Consider

$$V(x) = \int_0^{x_1} h_1(y)dy + \frac{1}{2}x_2^2$$

Clearly $V(0) = 0$, $V(x) > 0, \forall x \neq 0$ and $V(x) \to \infty$ as $|x| \to \infty$.

$$\begin{aligned}\dot{V}(x) &= h_1(x_1)\dot{x}_1 + x_2\dot{x}_2 \\ &= h_1(x_1)x_2 - x_2h_1(x_1) - x_2h_2(x_2) \\ &= -x_2h_2(x_2) \leq 0\end{aligned}$$

i.e, negative semidefinite.

Is it GAS?

$$S = \{x \in \mathbb{R}^2 : \dot{V}(x) = 0\} = \{x \in \mathbb{R}^2 : x_2 = 0\}$$

Let $x(t)$ be a solution that belongs identically to S:

$$x_2(t) = 0 \Rightarrow \dot{x}_2 = 0 \Rightarrow x_1 = 0$$

Therefore, the only solution that can stay identically in S is the trivial solution $x(t) = 0$. Thus, $x = 0$ is GAS.

Remark: LaSalle's theorem extends Lyapunov theorem in three ways besides the one that we saw above:

1. Can be used in cases where the system has an equilibrium set,
2. It can give an estimate of the region of attraction which is not necessarily of the form $\Omega_c = \{x : V(x) \leq c\}$,
3. $V(x)$ does not have to be positive definite.

Example

$$\dot{y} = ay + u$$

where $y \in \mathbb{R}$, a is an unknown but constant parameter and u is the input signal. Consider the following adaptive control law

$$u = -ky, \quad \dot{k} = \gamma y^2, \quad \gamma > 0$$

Let $x_1 = y$ and $x_2 = k$ then the closed loop system is given by

$$\begin{aligned} \dot{x}_1 &= -(x_2 - a)x_1 \\ \dot{x}_2 &= \gamma x_1^2 \end{aligned}$$

We have an equilibrium set $S_e = \{x \in \mathbb{R}^2 : x_1 = 0\}$.

Goal: Show that $y \to 0$, that is, x approaches S_e as $t \to \infty$.

Goal: Show that $y \to 0$, that is, x approaches S_e as $t \to \infty$.

Consider the Lyapunov function candidate

$$V(x) = \frac{1}{2}x_1^2 + \frac{1}{2\gamma}(x_2 - b)^2$$

where $b > a$ but we do not know its value explicitly.

$$\begin{aligned}\dot{V}(x) &= x_1\dot{x}_1 + \frac{1}{\gamma}(x_2 - b)\dot{x}_2 \\ &= -(x_2 - a)x_1^2 + \frac{1}{\gamma}(x_2 - b)\gamma x_1^2 \\ &= -(b - a)x_1^2 \leq 0.\end{aligned}$$

Let $\Omega_x = \{x \in \mathbb{R}^2 : V(x) \leq c\}$ where c can be chosen large enough since V is radially unbounded. Let $E = \{x \in \mathbb{R}^2 : x_1 = 0\}$. By LaSalle's theorem with $r = r_c$ we conclude that $x(t)$ approaches $M = E$ since E is an invariant set.

Linear systems and Linearization

Consider the time-invariant linear system

$$\dot{x} = Ax$$

Theorem 4.6

A matrix A is *Hurwitz*, that is, $Re(\lambda_i) < 0$ for all eigenvalues of A, if and only if, for any given $Q = Q^T > 0$ there exists a $P = P^T > 0$ that satisfies the *Lyapunov equation*

$$PA + A^T P = -Q \tag{2}$$

Moreover, if A is Hurwitz, then P is unique solution of (2).

How can we use this result to study the stability of $\dot{x} = f(x)$?

Linear systems and Linearization

Goal: Study the stability of $\dot{x} = f(x)$ using linearization

$$\dot{x} = f(x) \quad \Leftrightarrow \quad \dot{x} = Ax + g(x)$$

where $A = \frac{\partial f(x)}{\partial x}\Big|_{x=0}$ and $g(x) = f(x) - Ax$.
In particular,

$$g_i(x) = f_i(x) - \frac{\partial f_i(0)}{\partial x}$$

By the mean value theorem

$$f_i(x) = f_i(0) + \frac{\partial f_i(z_i)}{\partial x} x,$$

where z_i is a point on the line segment connecting x to the origin.
Therefore

$$g_i(x) = \left[\frac{\partial f_i(z_i)}{\partial x}(z_i) - \frac{\partial f_i(0)}{\partial x}(z_i)\right] x$$

and satisfies

$$|g_i(x)| \leq \left\| \frac{\partial f_i(z_i)}{\partial x}(z_i) - \frac{\partial f_i(0)}{\partial x}(z_i) \right\| \|x\|$$

By continuity of $\frac{\partial f}{\partial x}$ we can conclude that

$$\frac{\|g(x)\|}{\|x\|} \to 0 \text{ as } \|x\| \to 0$$

Lyapunov's indirect method

Theorem 4.7 - Lyapunov's indirect method

Let $x = 0$ be an equilibrium point for the nonlinear system

$$\dot{x} = f(x)$$

where $f : D \to \mathbb{R}^n$ is C^1 and D is a neighborhood of the origin. Let

$$A = \left.\frac{\partial f(x)}{\partial x}\right|_{x=0}$$

Then,

1. The origin is asymptotically stable if $Re(\lambda_i) < 0$ for all eigenvalues of A.
2. The origin is unstable if $Re(\lambda_i) > 0$ for one or more of the eigenvalues of A.

Remark that if $\lambda_i = 0, \forall i$ we cannot say nothing about it. For example $\dot{x} = ax^3 \Rightarrow A = \left.\frac{\partial f}{\partial x}\right|_{x=0} = 3ax^3\big|_{x=0} = 0$. For $a = 0, \dot{x} = 0$, which implies that the origin is stable. Taking $V = \frac{1}{2}x^2$ we have $\dot{V} = ax^4$ and if $a < 0$ the origin is asymptotically stable, if $a > 0$ then it is unstable.

Proof

Starting with the proof of (1). Consider the following Lyapunov function candidate

$$V(x) = x^T P x$$

The derivative of $V(x)$ along the trajectories is given by

$$\begin{aligned}\dot{V}(x) &= x^T P f(x) + f(x)^T P x \\ &= x^T P[Ax + g(x)] + [x^T A^T + g^T(x)]Px \\ &= x^T (PA + A^T P)x + 2x^T P g(x) \\ &= -x^T Q x + 2xPg(x)\end{aligned}$$

Since A is Hurwitz, $x^T QX > 0$. Regarding the other term, note that

$$\frac{\|g(x)\|_2}{\|x\|_2} \to 0$$

as $\|x\|_2 \to 0$.

Proof

Therefore $\forall \gamma > 0 \ \exists r > 0$ such that

$$\frac{\|g(x)\|_2}{\|x\|_2} < \gamma, \quad \forall \, \|x\|_2 < r$$

which implies that

$$\|g(x)\|_2 < \gamma \|x\|_2, \quad \forall \, \|x\|_2 < r$$

Hence,

$$\dot{V}(x) < -x^T Q x + 2\gamma \|P\|_2 \|x\|_2^2, \quad \forall \|x\|_2 < r$$

Using the fact that

$$0 < \lambda_{\min}(Q) \|x\|_2^2 \leq x^T Q x \leq \lambda_{\max}(Q) \|x\|_2^2$$

it follows that

$$\dot{V}(x) < -[\lambda_{\min}(Q) - 2\gamma \|P\|_2] \|x\|_2^2, \quad \forall \|x\|_2 < r$$

Choosing γ such that $\lambda_{\min}(Q) > 2\gamma \|P\|_2$, that is,

$$\gamma < \frac{\lambda_{\min}(Q)}{2\|P\|_2} \quad \Rightarrow \quad \dot{V}(x) < 0$$

and therefore $x = 0$ is asymptotically stable.

Comparison Functions

Definition

A continuous function $\alpha : [0, a) \to [0, \infty)$ is said to belong to class $\mathcal{K}$ if it is strictly increasing and $\alpha(0) = 0$. It is of class $\mathcal{K}_\infty$ if $a = \infty$ and $\alpha(r) \to \infty$ as $r \to \infty$.

Definition

A continuous function $\beta : [0, a) \times [0, \infty) \to [0, \infty)$ is said to belong to class $\mathcal{KL}$ if, for each fixed t, the mapping $\beta(r, t)$ belong to class K with respect to r and, for each fixed r, the mapping $\beta(r, t)$ is decreasing with respect to t and $\beta(r, t) \to 0$ as $t \to \infty$.

Examples

- Class $\mathcal{K}_\infty$

$$\alpha(r) = r^c, \quad c > 0$$

$$\alpha'(r) = cr^{c-1} > 0, \quad \lim_{r \to \infty} \alpha(r) = \infty$$

- Class $\mathcal{K}$

$$\alpha(r) = \tan^{-1}(r)$$

- Class $\mathcal{KL}$

  1. $$\beta(r,t) = r^c e^{-t}, \quad c > 0$$

  2. $$\beta(r,t) = \frac{r}{ktr+1}, \quad k > 0,$$

$$\frac{\partial \beta}{\partial r} = \frac{1}{(ktr+1)^2} > 0, \quad \frac{\partial \beta}{\partial t} = \frac{-kr^2}{(ktr+1)^2} < 0$$

so $\beta(r,t) \to 0$ as $t \to \infty$.

Properties

Lemma (Properties)

Let $\alpha_1, \alpha_2 \in \mathcal{K}$ *on* $[0, a)$*,* $\alpha_3, \alpha_4 \in \mathcal{K}_\infty$ *and* $\beta \in \mathcal{KL}$ *and* α_i^{-1} *denotes the inverse of* α_i*:*

1. α_1^{-1} *is defined on* $[0, \alpha_1(a))$ *and belongs to class* $\mathcal{K}$
2. α_3^{-1} *is defined on* $[0, \infty)$ *and belongs to class* $\mathcal{K}_\infty$
3. $\alpha_1 \circ \alpha_2$ *belongs to class* $\mathcal{K}$
4. $\alpha_3 \circ \alpha_4$ *belongs to class* $\mathcal{K}_\infty$
5. $\sigma(r, t) = \alpha_1(\beta(\alpha_2(r), t))$ *belongs to class* $\mathcal{KL}$

Comparison Functions

Lemma 4.3

Let $V : D \to \mathbb{R}$ be a continuous positive definite function defined on a domain $D \subset \mathbb{R}^n$ that contains the origin. Let $B_r \subset D$ for some $r > 0$. Then, there exist $\alpha_1, \alpha_2 \in \mathcal{K}$ defined on $[0, a]$ such that

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|), \ \forall x \in B_r$$

If $D = \mathbb{R}^n$, α_1 and α_2 are defined on $[0, \infty)$. Moreover, if $V(x)$ is radially unbounded, then $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$.

Example: Consider a quadratic positive definite function

$$V(x) = x^T P x$$

We know that

$$\lambda_{\min}(P)\|x\|_2^2 \leq x^T P x \leq \lambda_{\max}(P)\|x\|_2^2$$

and so we can define $\alpha_1(r) = \lambda_{\min}(P) r^2$ and $\alpha_2(r) = \lambda_{\max}(P) r^2$

Comparison Functions

Lemma 4.4

Consider the scalar autonomous differential equation

$$\dot{y} = -\alpha(y), \quad y(t_0) = y_0$$

where α is a locally Lipschitz class $\mathcal{K}$ function defined on $[0, a)$. Then, for all $0 \leq y_0 < a$, the solution is unique and defined for all $t \geq t_0$. Moreover,

$$y(t) = \beta(y_0, t - t_0)$$

where $\beta \in \mathcal{KL}$ is defined on $[0, a) \times [0, \infty)$.

a) $\dot{y} = -ky,\ k > 0$ the solution is given by $y(t) = y_0 e^{-k(t-t_0)}$ and we can define $\beta(r, t) = re^{-kt}$

b) $\dot{y} = -ky^2,\ k > 0$ the solution is given by $y(t) = \frac{y_0}{ky_0(t-t_0)+1}$ and we can define $\beta(r, t) = \frac{r}{krt+1}$

Alternative proof of Lyapunov Theorem (Theorem 4.1)

In the proof, given an $\epsilon > 0$, we choose $r \leq \epsilon$ such that $B_r \subset D$. Now, we would like to choose β and δ such that

$$B_\delta \subset \Omega_\beta \subset B_r$$

But $V(x)$ satisfies

$$\alpha_1(\|x\|) \leq V(x) \leq \alpha_2(\|x\|)$$

Therefore select

$$\beta \leq \alpha_1(r)$$

Why?

$$V(x) \leq \beta \leq \alpha_1(r) \Rightarrow \alpha_1(\|x\|) \leq \alpha_1(r) \Leftrightarrow \|x\| \leq r$$

so we conclude that $\forall x \in \Omega_\beta \Rightarrow x \in B_r$.
Also select

$$\delta \leq \alpha_2^{-1}(\beta)$$

Why?

$$\|x\| \leq \delta \Rightarrow V(x) \leq \alpha_2(\delta) \leq \beta \Rightarrow B_\delta \subset \Omega_\beta$$

Alternative proof of Lyapunov Theorem

To prove asymptotic stability note that

$$\dot{V}(x) < 0 \Rightarrow \exists \alpha_3, \alpha_4 \in \mathcal{K} : \alpha_3(\|x\|) \leq -\dot{V}(x) \leq \alpha_4(\|x\|)$$

In particular we have that $\dot{V}(x) \leq -\alpha_3(\|x\|)$. Hence

$$\dot{V} \leq -\alpha_3(\alpha_2^{-1}(V))$$

Using the comparison lemma it follows that $V \leq y$ for

$$\dot{y} = -\alpha_3(\alpha_2^{-1}(y)), \quad y(0) = V(x(0))$$

From Lemma 4.2 it follows that $\alpha_3 \circ \alpha_2^{-1} \in \mathcal{K}$ and from Lemma 4.4 we conclude that

$$y(t) = \beta(y(0), t),\ \ \beta \in \mathcal{KL}$$

In conclusion $V(x(t)) \leq \beta(V(x(0)), t)$, which shows that V is bounded and $V \rightarrow 0$ as $t \rightarrow \infty$.

Alternative proof of Lyapunov Theorem

In fact we can estimate a bound for $x(t)$

From

$$V(x(t)) \leq V(x(0)) \Rightarrow \alpha_1(\|x(t)\|) \leq V(x(t)) \leq V(x(0)) \leq \alpha_2(\|x(0)\|)$$

and therefore,

$$\|x(t)\| \leq \alpha_1^{-1}(\alpha_2(\|x(0)\|))$$

where $\alpha_1^{-1} \circ \alpha_2 \in \mathcal{K}$

Similarly, from

$$V(x(t)) \leq \beta(V(x(0)), t) \Rightarrow \alpha_1(\|x\|) \leq V(x(t)) \leq \beta(V(x(0)), t) \leq \beta(\alpha_2(\|x(0)\|), t)$$

we obtain

$$\|x\| \leq \alpha_1^{-1}(\beta(\alpha_2(\|x(0)\|), t)) := \sigma(\|x(0)\|, t)$$

with $\sigma \in \mathcal{KL}$.

Nonautonomous Systems

Suppose we would like to analyze the stability behavior of the *solution* $\bar{y}(t)$ of the system

$$\dot{y} = g(t, y)$$

How can we do this?

Define

$$x(t) = y(t) - \bar{y}(t)$$

Then, we have

$$\dot{x} = g(t, y) - \dot{\bar{y}} = g(t, x(t) + \bar{y}(t)) - \dot{\bar{y}}(t) := f(t, x)$$

and therefore we obtain the nonautonomous system

$$\dot{x} = f(t, x)$$

with $f(t, 0) = 0, \ \forall t \geq 0$.

This means that we only need to study the stability of the equilibrium point $x = 0$.

Nonautonomous Systems

Consider the nonautonomous system

$$\dot{x} = f(t, x)$$

Definition

The origin is an equilibrium point at $t = 0$ if

$$f(t, 0) = 0, \quad \forall t \geq 0.$$

Nonautonomous Systems

Consider the nonautonomous system

$$\dot{x} = f(t, x)$$

Definition

The equilibrium point $x = 0$ of the nonautonomous system is

- Stable if $\forall \epsilon > 0 \; \exists \delta = \delta(\epsilon, t_0) > 0 : \|x(t_0)\| < \delta \Rightarrow \|x(t)\| < \epsilon, \; \forall t \geq t_0 \geq 0$
- Uniformly Stable (US) if δ is independent of t_0
- Asymptotically Stable if it is stable and $\exists c = c(t_0) : x(t) \to 0$ as $t \to \infty$ $\forall \|x(t_0)\| < c$
- Uniformly Asymptotically Stable (UAS) if c is independent of t_0 and the convergence is uniformly in t_0, that is

$$\forall \eta > 0 \; \exists T = T(\eta) > 0 : \|x(t)\| < \eta, \quad \forall t \geq t_0 + \tau(\eta), \forall \|x(t_0)\| < c$$

- Globally Uniformly Asymptotic Stable (GUAS) if $\delta(\epsilon)$ can be chosen to satisfies $\lim_{\epsilon \to \infty} \delta(\epsilon) = \infty$ and

$$\forall (\eta, c) \; \exists T = T(\eta, c) > 0 : \|x(t)\| < \eta, \quad \forall t \geq t_0 + T(\eta, c), \forall \|x(t_0)\| < c$$

Nonautonomous Systems

The following lemma make the above definitions more clear.

Lemma 4.5

The equilibrium point $x = 0$ of the nonautonomous system is

- US $\Leftrightarrow$ there exist $\alpha \in \mathcal{K}$ and $c > 0$ (independent of t_0) such that

$$\|x(t)\| \leq \alpha(\|x(t_0)\|), \quad \forall t \geq t_0 \geq 0, \forall \|x(t_0)\| < c$$

- UAS $\Leftrightarrow$ there exist $\beta \in \mathcal{KL}$ and $c > 0$ (independent of t_0) such that

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \quad \forall t \geq t_0 \geq 0, \forall \|x(t_0)\| < c$$

- GUAS $\Leftrightarrow$ the inequality in UAS holds for all $x(t_0)$, that is $c = \infty$

Definition

The equilibrium point $x = 0$ is exponentially stable if there exists positive constants $c,\ k$ and λ such that

$$\|x(t)\| \leq k\|x(t_0)\|e^{-\lambda(t-t_0)}, \quad \forall \|x(t_0\| < c$$

and globally exponentially stable if this inequality is satisfied $\forall x(t_0)$

Theorem 4.8

Let $x = 0$ be an equilibrium point that belongs to $D \subset \mathbb{R}^n$. Let $V : [0, \infty) \times D \to \mathbb{R}$ be a C^1 function such that

$$W_1(x) \leq V(t, x) \leq W_2(x)$$

and

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq 0,\ \forall t \geq 0\ \forall x \in D$$

where $W_1(x)$, $W_2(x)$ are continuous positive functions on D. Then, $x = 0$ is uniformly stable.

Proof

Choose $r > 0$ and $c > 0$ such that $B_r \subset D$ and
$0 < c < \min_{\|x\|=r} W_1(x) \Rightarrow \{x \in B_r : W_1(x) \leq c\} \subset B_r$

Define a time-dependent set $\Omega_{t,c}$ such that

$$\{x \in B_r : W_2(x) \leq c\} \subset \Omega_{t,c} = \{x \in B_r : V(t,x) \leq c\} \subset \{x \in B_r : W_1(x) \leq c\}$$

Hence the solution is bounded and defined for all $t \geq t_0$. Moreover, since $\dot{V} \leq 0$ we have that $V(t, x(t)) \leq V(t_0, x(t_0)), \forall t \geq t_0$.
Also

$$\exists_{\alpha_1,\alpha_2 \in \mathcal{K}} : \alpha_1(\|x\|) \leq W_1(x) \leq V(t,x) \leq W_2(x) \leq \alpha_2(\|x\|)$$

and therefore,

$$\|x(t)\| \leq \alpha_1^{-1}(V(t,x(t))) \leq \alpha_1^{-1}(V(t_0,x(t_0))) \leq \alpha_1^{-1}(\alpha_2(\|x(t_0)\|))$$

Thus, we can conclude that $x = 0$ is US since $\alpha_1^{-1} \circ \alpha_2 \in \mathcal{K}$.

Theorem 4.9

Same assumptions as Theorem 4.8 but with

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -W_3(x), \quad \forall t \geq 0 \; \forall x \in D$$

where $W_3(x)$ is a continuous positive definite function on D. Then, $x = 0$ is UAS. Moreover, if r and c are such that

$$B_r = \{\|x\| \leq r\} \subset D$$

and

$$c < \min_{\|x\| \leq r} W_1(x)$$

Then

$$\forall x_0 \in \{x \in B_r : W_2(x) \leq c\},\; \|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0),\; \forall t \geq t_0 \geq 0$$

for some $\beta \in \mathcal{KL}$.
If in addition $D = \mathbb{R}^n$ and $W_1(x)$ is radially unbounded then $x = 0$ is GUAS.

Proof

$$\dot{V}(t,x) = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -W_3(x) \leq -\alpha_3(\|x\|),$$

for some $\alpha_3 \in \mathcal{K}$,

$$V \leq \alpha_2(\|x\|) \Leftrightarrow \alpha_2^{-1}(V) \leq \|x\| \Leftrightarrow \alpha_3(\alpha_2^{-1}(V)) \leq \alpha_3(\|x\|)$$

Thus,

$$\dot{V} \leq -\alpha_3(\alpha_2^{-1}(V)) := -\alpha(V), \qquad (3)$$

where $\alpha \in \mathcal{K}$.

Without loss of generality α is locally Lipschitz. Why? If not we can always choose $\gamma \in \mathcal{K}$, with γ Lipschitz such that $\alpha(r) \geq \gamma(r)$. Then $\dot{V} \leq -\gamma(V)$.

Proof

For example, suppose $\alpha(r) = \sqrt{r}$ is a class $\mathcal{K}$ function, but not locally Lipschitz at $r = 0$. Define

$$\gamma(r) = \begin{cases} r, & r < 1 \\ \sqrt{r}, & r \geq 1 \end{cases}$$

and so, $\gamma \in \mathcal{K}$ locally Lipschitz and $\alpha(r) \geq \gamma(r), \forall r \geq 0$.

Returning to (3) and resorting to the comparison lemma,

$$\dot{y} = -\alpha(y),\ y(t_0) = V(t_0, x(t_0)) \geq 0 \Rightarrow V(t, x(t)) \leq y(t),\ \forall t \geq t_0$$

By Lemma 4.4, $\exists \sigma \in \mathcal{KL}$ such that

$$V(t, x(t)) \leq \sigma(V(t_0, x(t_0)), t - t_0)$$

In conclusion, $\forall x(t_0) \in \{x \in B_r : W_2(x) \leq c\}$ we have

$$\|x\| \leq \alpha_1^{-1}(V(t, x(t))) \leq \sigma(V(t_0, x(t_0)), t - t_0) \leq \sigma(\alpha_2(\|x(t_0)\|), t - t_0) =: \beta(\|x(t_0)\|, t - t_0)$$

Definition

- $V(t,x)$ is said to be positive semidefinite if $V(t,x) \geq 0$
- $V(t,x)$ is said to be positive definite if $V(t,x) \geq W_1(x)$, for some $W_1(x)$ positive definite function.
- $V(t,x)$ is radially unbounded if $W_1(x)$ is radially unbounded.

Theorem 4.10

Let $x = 0$ be an equilibrium point for the nonautonomous system $\dot{x} = f(t,x)$ that belongs to some $D \subset \mathbb{R}^n$. Let $V : [0,\infty) \times D \to \mathbb{R}$ be a C^1 function such that

$$k_1\|x\|^a \leq V(t,x) \leq k_2\|x\|^a$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t,x) \leq -k_3\|x\|^a, \ \forall t \geq 0, \ \forall x \in D$$

where k_1, k_2, k_3 and a are positive constants. Then $x = 0$ is exponentially stable. If the assumption holds globally. Then $x = 0$ is GES.

proof

$$\dot{V} \leq -\frac{k_3}{k_2} V \Rightarrow V(t, x(t)) \leq V(t_0, x(t_0)) e^{-\frac{k_3}{k_2}(t-t_0)}$$

Hence,

$$\|x\| \leq \left[\frac{V(t,x)}{k_1}\right]^{1/a} \leq \left[\frac{V(t_0,x(t_0)) e^{-\frac{k_3}{k_2}(t-t_0)}}{k_1}\right]^{1/a}$$

$$\leq \left[\frac{k_2 \|x(t_0)\|^a e^{-\frac{k_3}{k_2}(t-t_0)}}{k_1}\right]^{1/a} \leq \left[\frac{k_2}{k_1}\right]^a \|x(t_0)\| e^{-\frac{k_3}{k_2 a}(t-t_0)}$$

Examples

Example 1

$$\dot{x} = -(1 + g(t))x^3$$

where $x \in \mathbb{R}^2$, g is C^0 (continuous) and $g(t) \geq 0, \forall t \geq 0$.

Consider

$$V = \frac{1}{2}x^2$$

and so,

$$\dot{V} = -(1 + g(t))x^4 \leq -x^4$$

Then, $W_1(x) = W_2(x) = V(x)$ and $W_3(x) = x^4$, thus $x = 0$ is GUAS.

Note we cannot conclude exponential because is not the same a.

Example 2

$$\dot{x}_1 = -x_1 - g(t)x_2$$
$$\dot{x}_2 = x_1 - x_2$$

where g is C^1, $0 \leq g(t) \leq k$ and $\dot{g}(t) \leq g(t)$

Consider

$$V(t,x) = x_1^2 + (1 + g(t))x_2^2$$

Note that,

$$x_1^2 + x_2^2 \leq V(t,x) \leq x_1^2 + (1+k)x_2^2$$

Thus $V(t,x)$ is positive definite and radially unbounded.

$$\begin{aligned}
\dot{V}(t,x) &= 2x_1\dot{x}_1 + 2x_2(1+g(t))\dot{x}_2 + \dot{g}x_2^2 \\
&= -2x_1^2 - 2gx_1x_2 + 2x_2x_1 - 2x_2^2 + 2x_2gx_1 - 2x_2^2g + \dot{g}x_2^2 \\
&= -2x_1^2 + 2x_2x_1 - [2 + 2g - \dot{g}]x_2^2 \\
&\leq -2x_1^2 + 2x_1x_2 - 2x_2^2 \\
&= -\begin{bmatrix} x_1 & x_2 \end{bmatrix} \begin{bmatrix} 2 & -1 \\ -1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} =: -x^TQx
\end{aligned}$$

Therefore, $W_1,\ W_2$ and W_3 are positive definite quadratic functions (a=2) ($\lambda_{\min}(P)x^Tx \leq x^TPx \leq \lambda_{\max}(P)x^Tx$) and so we conclude that $x = 0$ is GES.

Linear time-varying systems and linearization

$$\dot{x}(t) = A(t)x(t)$$

the solution can be represented as

$$x(t) = \Phi(t, t_0)x(t_0)$$

From linear system theory...

Theorem 4.11

The equilibrium point $x = 0$ of the linear system $\dot{x}(t) = A(t)x(t)$ is GUAS if and only if

$$\|\Phi(t, t_0)\| \leq ke^{-\lambda(t-t_0)}, \ \forall t \geq t_0 \geq 0$$

for some positive constants k and λ.

Remark that

1. for linear systems GUAS $\Leftrightarrow$ Exponential stability
2. for linear time-varying system, GUAS cannot be characterized by the location of the eigenvalues of A.

Example

$$A(t) = \left[\begin{array}{cc} -1 + 1.5\cos^2 t & 1 - 1.5 \sin t \cos t \\ 1 - 1.5 \sin t \cos t & -1 + 1.5\sin^2 t \end{array} \right]$$

For each t, $\lambda_i(A(t)) = -0.25 \pm 0.25\sqrt{7}j$. Yet, the origin is unstable. $\Phi(t,0)$ is given by

$$\Phi\left(t,0\right) = \left[\begin{array}{cc} e^{0.5t}\cos t & e^{-t}\sin t \\ -e^{0.5t}\sin t & e^{-t}\cos t \end{array} \right]$$

Theorem 4.12

Let $x = 0$ be a exponential stable equilibrium point of the linear system $\dot{x}(t) = A(t)x(t)$. Suppose $A(t)$ is continuous and bounded. Let $Q(t) = Q^T(t) > 0$ be continuous and bounded. Then, there is a $P(t) = P^T(t) \in C^2$ bounded matrix that satisfies

$$-\dot{P}(t) = P(t)A(t) + A^T(t)P(t) + Q(t).$$

Furthermore, $V(t, x) = x^T P(t)x$ is a Lyapunov function that satisfies

$$c_1 \|x\|_2^2 \leq x^T P(t)x \leq c_2 \|x\|_2^2$$

and

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} A(t)x \leq -c_3 \|x\|_2^2$$

We are now ready for the following theorem:

Theorem 4.13/4.15

Let $x = 0$ be an equilibrium point for the nonlinear system

$$\dot{x} = f(t, x) \tag{4}$$

where $f : [0, \infty) \times D \to \mathbb{R}^n$ is C^1, $D = \{x \in \mathbb{R}^2 : \|x\|_2 < r\}$, and the Jacobian matrix $\frac{\partial f}{\partial x}$ is bounded and Lipschitz on D, uniformly in t. Let

$$A(t) = \frac{\partial f}{\partial x}(t, x)\bigg|_{x=0}$$

Then, the origin is an exponentially stable equilibrium point of (4) if and only if it is an exponentially stable equilibrium point for the linear system $\dot{x} = A(t)x$.

Proof

Since the Jacobian is bounded

$$\|\frac{\partial f_i}{\partial x}(t, x_1) - \frac{\partial f_i}{\partial x}(t, x_2)\|_2 \leq L_1\|x_1 - x_2\|_2, \ \forall x_1, x_2 \in D$$

By the mean value theorem

$$f_i(t, x) = f_i(t, 0) + \frac{\partial f_i}{\partial x}(t, z_i)x$$

where $f_i(t, 0) = 0$ and z_i is a point of the line segment connecting x to the origin. Thus,

$$f_i(t, x) = \frac{\partial f_i}{\partial x}(t, 0)x + \left[\frac{\partial f_i}{\partial x}(t, z_i) - \frac{\partial f_i}{\partial x}(t, 0)\right] x$$

Hence,

$$f(x, t) = A(t)x + g(t, x)$$

where $A(t) = \frac{\partial f_i}{\partial x}(t, 0)$ and $g(t, x) = \left[\frac{\partial f_i}{\partial x}(t, z_i) - \frac{\partial f_i}{\partial x}(t, 0)\right] x$. Thus, we have

$$\|g(t, x)\|_2 \leq \left(\sum_{i=1}^{n} \left\|\frac{\partial f_i}{\partial x}(t, z_i) - \frac{\partial f_i}{\partial x}(t, 0)\right\|_2^2\right)^{1/2} \|x\|_2 \leq L\|x\|_2$$

where $L = \sqrt{n}L_1$.

Proof

$$V(t,x) = x^T P(t) x$$

$$\begin{aligned}
\dot{V}(t,x) &= x^T P(t) f(t,x) + f^T(t,x) P(t) x + x^T \dot{P}(t) x \\
&= x^T [P(t)A(t) + A^T(t)P(t) + \dot{P}(t)] x + 2x^T P(t) g(t,x) \\
&= -x^T Q(t) x + 2x^T P(t) g(t,x) \\
&\leq -c_3 \|x\|_2^2 + 2c_2 L \|x\|_2^3 \\
&\leq -(c_3 - 2c_2 L\rho) \|x\|_2^2, \ \forall \|x\|_2 < \rho
\end{aligned}$$

Choosing $\rho < \min\{r, \frac{c_3}{2c_2 L}\}$, implies that $\dot{V}(t,x) < 0, \ \forall \|x\|_2 < \rho$ and thus $x = 0$ is exponentially stable.

Converse Theorems

Theorem 4.16

Let $x = 0$ be an equilibrium point for the nonlinear system

$$\dot{x} = f(t, x)$$

where $f : [0, \infty) \times D \to \mathbb{R}^n$ is C^1, $D = \{x \in \mathbb{R}^2 : \|x\| < r\}$ and the Jacobian matrix $\frac{\partial f}{\partial x}$ is bounded on D, uniformly in t. Let $\beta \in \mathcal{KL}$ and $r_0 > 0$ and that $\beta(r_0, 0) < r$. Let $D_0 = \{x \in \mathbb{R}^n : \|x\| < r_0\}$. Suppose that

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0), \quad \forall x(t_0) \in D_0 \ \forall t \geq t_0 \geq 0.$$

Then, there is a C^1 function $V : [0, \infty) \times D_0 \to \mathbb{R}$ that satisfies

$$\alpha_1(\|x\|) \leq V(t, x) \leq \alpha_2(\|x\|)$$
$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \leq -\alpha_3(\|x\|)$$
$$\left\| \frac{\partial V}{\partial x} \right\| \leq \alpha_4(\|x\|),$$

where $\alpha_1, \alpha_2, \alpha_3, \alpha_4$ are class $\mathcal{K}$ functions defined on $[0, r_0]$. If the system is autonomous, V can be chosen independently of t.
In particular, for $\beta(r, t) = kre^{-\lambda t}$ we have $\alpha_i = c_i r^2, i = 1, 2, 3$, $\alpha_4 = c_4 r$ together with

$$V(t, x) = \int_t^{t+\delta} \Phi^T(\tau; t, x) \Phi(\tau; t, x) d\tau$$

where $\Phi(\tau; t, x)$ denotes the solution of the system that starts at (t, x).

Boundedness and Ultimate Boundedness

Until now we have used Lyapunov theory to study the behavior of the system about the equilibrium point.

What happens when the system does not have any equilibrium point?
We will see that Lyapunov analysis can be used to show boundedness of the solution of the state equation.

Example

$$\dot{x} = -x + \delta \sin t,\ x(t_0) = a,\ a > \delta > 0$$

There are no equilibrium points!

Nevertheless, with

$$V(x) = \frac{1}{2}$$

we obtain

$$\dot{V} = -x^2 + x\delta \sin t \leq -x^2 + \delta|x|.$$

Note that $\dot{V} < 0, \forall |x| > \delta$, which means that the set $\{x \in \mathbb{R} : V(x) \leq c\}$ with $c > \frac{\delta^2}{2}$ is an invariant set because $V = c \rightarrow \dot{V} < 0$.

Hence, the solutions are *uniformly bounded*.

Boundedness and Ultimate Boundedness

Moreover

$$\forall \epsilon > 0 : \frac{\delta^2}{2} < \epsilon < c \quad \longrightarrow \quad \dot{V} < -\gamma$$

for some $\gamma > 0$ in the set $\epsilon \leq V \leq c$ which shows that V will reach in finite time ϵ and the solution enter the set $\{V \leq \epsilon\}$.
Thus, we can conclude that the solution is *uniformly ultimately bounded* with *ultimate bound* $|x| \leq \sqrt{2\epsilon}$.

Definition

The solutions of $\dot{x} = f(t, x)$ are

- Uniformly Bounded (UB) if there exists a $c > 0$, independent of $t_0 \geq 0$ such that

$$\forall a \in (0, c)\ \exists \beta = \beta(a) > 0 : \|x(t_0)\| \leq a \Rightarrow \|x(t)\| \leq \beta,\ \forall t \geq t_0$$

- Globally Uniformly Bounded (GUB) if a can be arbitrarily large
- Uniformly Ultimately Bounded (UUB) with ultimate bound b, if there exists $b > 0$, $c > 0$ (independent of t_0) such that

$$\forall a \in (0, c)\ \exists T = T(a, b) \geq 0 : \|x(t_0)\| \leq a \Rightarrow \|x(t)\| \leq b,\ \forall t \geq t_0 + T,$$

- Globally Uniformly Ultimately Bounded (GUUB) if a can be arbitrarily large.

Theorem 4.18 (Global)

Let $V : [0, \infty) \times \mathbb{R}^n \to \mathbb{R}$ be a C^1 function such that

$$\alpha_1(\|x\|) \le V(t, x) \le \alpha_2(\|x\|)$$

$$\frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x) \le -W_3(x), \ \ \forall \|x\| \ge \mu > 0$$

$\forall t \ge 0$ and $\forall x \in \mathbb{R}^n$, where $\alpha_1, \alpha_2 \in \mathcal{K}_\infty$ and $W_3(x) > 0$.

Then, there exists a $\beta \in \mathcal{KL}$ and $T \ge 0$ such that

$$\|x(t)\| \le \beta(\|x(t_0)\|, t - t_0), \forall t_0 \le t \le t_0 + T$$

and

$$\|x(t)\| \le \alpha_1^{-1}(\alpha_2(\mu)), \forall t \ge t_0 + T$$

Example - Mass-spring system

$$m\ddot{y} + c\dot{y} + ky + ka^2y^3 = F = A\cos(wt)$$

Taking

$$\begin{aligned} x_1 &= y \\ x_2 &= \dot{y} \end{aligned}$$

and assuming certain numerical values we get

$$\begin{aligned} \dot{x}_1 &= x_2 \\ \dot{x}_2 &= -(1 + x_1^2)x_1 - x_2 + M\cos(wt) \end{aligned}$$

where $M \geq 0$ is proportional to A. Choosing

$$V = x^T \begin{bmatrix} 3/2 & 1/2 \\ 1/2 & 1 \end{bmatrix} x + 1/2 x_1^4$$

then

$$\begin{aligned} \dot{V} &= -x_1^2 - x_1^4 - x_2^2 + (x_1 + 2x_2)\, M\cos(wt) \\ &= -\|x\|^2 - x_1^4 + \begin{bmatrix} 1 & 2 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix} M\cos(wt) \\ &\leq -\|x\|^2 - x_1^4 + M\sqrt{5}\,\|x\| \\ &\leq -(1-\theta)\|x\|^2 - x_1^4 - \theta\|x\|^2 + M\sqrt{5}\,\|x\|, \;\; 0 < \theta < 1 \end{aligned}$$

Example - Mass-spring system

Therefore

$$\dot{V} \leq -(1-\theta)\|x\|^2 - x_1^4, \quad \forall \|x\| \geq \frac{M\sqrt{5}}{\theta}$$

which shows that the solutions are GUUB. To compute the ultimate bound, we have to find α_1, α_2.

$$V(x) \geq x^T P x \geq \lambda_{\min}(P)\|x\|^2 \quad \rightarrow \quad \alpha_1(r) = \lambda_{\min}(P) r^2$$

$$V(x) \leq x^T P x + \frac{1}{2}\|x\|^4 \leq \lambda_{\max}(P)\|x\|^2 + \frac{1}{2}\|x\|^4 \quad \rightarrow \quad \alpha_2(r) = \lambda_{\max}(P) r^2 + \frac{1}{2} r^4$$

Let $\mu = \frac{M\sqrt{5}}{\theta}$

$$\|x\| \leq \mu \Rightarrow V(x) \leq \alpha_2(\mu) = \epsilon$$

which ensures that

$$B_\mu \subset \Omega_\epsilon = \{x : V(x) \leq \epsilon\}$$

But

$$V(x) \leq \epsilon \Rightarrow \alpha_1(\|x\|) \leq \epsilon \Leftrightarrow \|x\| \leq \alpha_1^{-1}(\epsilon)$$

therefore

$$x \in \Omega_\epsilon \Rightarrow \|x\| \leq \alpha_1^{-1}(\alpha_2(\mu)) = \sqrt{\frac{\lambda_{\max}(P)\mu^2 + \frac{\mu^4}{2}}{\lambda_{\min}(P)}}$$

Input-to-state stability

For a linear time-invariant system

$$\dot{x} = Ax + Bu$$

with A Hurwitz. The solution is given by

$$x(t) = e^{A(t-t_0)}x(t_0) + \int_{t_0}^{t} e^{A(t-\tau)}Bu(\tau)d\tau$$

Using the bound

$$\|e^{A(t-t_0)}\| \leq ke^{-\lambda(t-t_0)}$$

we conclude that

$$\begin{aligned}
\|x(t)\| &\leq ke^{-\lambda(t-t_0)}\|x(t_0)\| + \int_{t_0}^{t} ke^{-\lambda(t-\tau)}\|B\|\|u(\tau)\|d\tau \\
&\leq ke^{-\lambda(t-t_0)}\|x(t_0)\| + \frac{k}{\lambda}(1 - e^{-\lambda(t-t_0)})\|B\| \sup_{t_0 \leq \tau \leq t} \|u(\tau)\| \\
&\leq ke^{-\lambda(t-t_0)}\|x(t_0)\| + \frac{k}{\lambda}\|B\| \sup_{t_0 \leq \tau \leq t} \|u(\tau)\|
\end{aligned}$$

Does this hold for a general nonlinear system?

$$\dot{x} = f(t, x, u)$$

And in what conditions? Is it sufficient to have GUAS of the unforced system? NO!

Input-to-state stability

Example

$$\dot{x} = -3x + (1 + 2x^2)u$$

when $u = 0$, the equilibrium point $x = 0$ is GAS. Yet, when $x(0) = 2$ and $u(t) = 1$, the solution is given by

$$x(t) = \frac{3 - e^t}{3 - 2e^t}$$

which is unbounded. It even has a finite escape time.

Definition

The system

$$\dot{x} = f(t, x, u)$$

is said to be input-to-state stable (ISS) if there exist $\beta \in \mathcal{KL}$, and $\gamma \in \mathcal{K}$ such that for any initial state $x(t_0)$ and any bounded input $u(t)$, the solution $x(t)$ exists for all $t \geq t_0$ and satisfies

$$\|x(t)\| \leq \beta(\|x(t_0)\|, t - t_0) + \gamma(\sup_{t_0 \leq \tau \leq t} \|u(\tau)\|)$$

Remarks:

1. For any bounded input $u(t)$, the state $x(t)$ will be bounded
2. As t increases, the state $x(t)$ will be ultimately bounded by a class $\mathcal{K}$ function of $\sup_{t\geq t_0}\|u(t)\|$
3. If $u(t)\to 0$ as $t\to\infty$ then $x(t)\to 0$ as $t\to\infty$
4. Since for $u(t)=0$, $\|x(t)\|\leq\beta(\|x(t_0)\|,t-t_0)\Leftrightarrow x=0$ of the unforced system is GUAS.

Theorem 4.19

Let $V:[0,\infty)\times\mathbb{R}^n\to\mathbb{R}$ be a C^1 function such that

$$\alpha_1(\|x\|)\leq V(t,x)\leq\alpha_2(\|x\|)$$

$$\frac{\partial V}{\partial t}+\frac{\partial V}{\partial x}f(t,x,u)\leq -W_3(x),\ \forall\|x\|\geq\rho(u)>0$$

$\forall(t,x,u)\in[0,\infty)\times\mathbb{R}^n\times\mathbb{R}^n$, $\alpha_1,\alpha_2\in\mathcal{K}_\infty$, $\rho\in\mathcal{K}$, $W_3(x)>0$ and C^0. Then the system $\dot{x}=f(t,x,u)$ is ISS with $\gamma=\alpha_1^{-1}\circ\alpha_2\circ\rho$.

Proof.

Apply Theorem 4.18. □

For autonomous systems the conditions are also necessary, that is, it is *if and only if*.

Lemma 4.6

Suppose that $f(t, x, u)$ is C^1 and globally Lipschitz in (x, u), uniformly in t. If the unforced system has a globally exponentially stable equilibrium point at $x = 0$, then the system $\dot{x} = f(t, x, u)$ is ISS

Proof.

From the converse theorem (Theorem 4.14) we conclude that the unforced system has a Lyapunov function $V(t, x)$. Thus,

$$\dot{V} = \frac{\partial V}{\partial t} + \frac{\partial V}{\partial x} f(t, x, 0) + \frac{\partial V}{\partial x} [f(t, x, u) - f(t, x, 0)]$$

and therefore

$$\dot{V} \leq -c_3 \|x\|^2 + c_4 \|x\| L \|u\| \leq -c_3(1 - \theta)\|x\|^2 - c_3 \theta \|x\|^2 + c_4 L \|x\| \|u\|$$

where $0 < \theta < 1$. Then,

$$\dot{V} \leq -c_3(1 - \theta)\|x\|^2, \ \forall \|x\| \geq \frac{c_4 L \|u\|}{c_3 \theta}$$

Apply Theorem 4.19 with

$$\alpha_1(r) = c_1 r^2, \ \alpha_2(r) = c_2 r^2, \ \rho(r) = \frac{c_4 L}{c_3 \theta} r$$

we conclude that the system is ISS with $\gamma(r) = \sqrt{\frac{c_2}{c_1}} \frac{c_4 L}{c_3 \theta} r$ □

Example

$$\dot{x} = -x^3 + u$$

Choose,

$$V = \frac{1}{2}x^2$$

Then

$$\begin{aligned}\dot{V} &= -x^4 + xu \\ &= -(1-\theta)x^4 - \theta x^4 + xu,\ 0 < \theta < 1 \\ &\leq -(1-\theta)x^4,\ \forall |x| \geq \left(\frac{|u|}{\theta}\right)^{1/3}\end{aligned}$$

which means that the system is ISS with $\gamma = \left(\frac{r}{\theta}\right)^{1/3}$ and $\gamma = \alpha_1^{-1} \circ \alpha_2 \circ \rho$.

Cascade System

Consider the following system

$$\dot{x}_1 = f_1(t, x_1, x_2) \tag{5}$$

where the equilibrium point $x_1 = 0$ of $\dot{x}_1 = f_1(t, x_1, 0)$, is GUAS in cascade with

$$\dot{x}_2 = f_1(t, x_2), \tag{6}$$

where the equilibrium point $x_2 = 0$ of $\dot{x}_2 = f_1(t, x_2)$, is GUAS.

Under what conditions will the origin $x = (x_1, x_2) = 0$ of the cascade system be GUAS?

Lemma 4.7

If system (5) is ISS with x_2 as input, then the origin of the cascade system is GUAS.

Proof

$$\|x_1(t)\| \le \beta_1(\|x_1(s)\|, t-s) + \gamma_1(\sup_{s\le\tau\le t} \|x_2(\tau)\|) \tag{7}$$

$$\|x_2(t)\| \le \beta_2(\|x_1(s)\|, t-s) \tag{8}$$

Taking (7) with $s = \frac{t+t_0}{2}$ yields

$$\|x_1(t)\| \le \beta_1(\|x_1(\frac{t+t_0}{2})\|, \frac{t-t_0}{2}) + \gamma_1(\sup_{\frac{t+t_0}{2}\le\tau\le t} \|x_2(\tau)\|)$$

but

$$\|x_1(\frac{t+t_0}{2})\| \le \beta_1(\|x_1(t_0)\|, \frac{t-t_0}{2}) + \gamma_1(\sup_{t_0\le\tau\le\frac{t+t_0}{2}} \|x_2(\tau)\|).$$

From (8)

$$\sup_{t_0\le\tau\le\frac{t+t_0}{2}} \|x_2(\tau)\| \le \beta_2(\|x_2(t_0)\|, 0)$$

$$\sup_{\frac{t+t_0}{2}\le\tau\le t} \|x_2(\tau)\| \le \beta_2(\|x_2(t_0)\|, \frac{t-t_0}{2})$$

Proof

Thus,

$$\begin{aligned}\|x_1(t)\| \le \beta_1(\beta_1(\|x_1(t_0)\|, \frac{t-t_0}{2})\\ + \gamma_1(\beta_2(\|x_2(t_0)\|, 0)), \frac{t-t_0}{2})\\ + \gamma_1(\beta_2(\|x_2(t_0)\|, \frac{t-t_0}{2}))\end{aligned}$$

Since $\|x(t)\| \le \|x_1(t)\| + \|x_2(t)\|$ and $\|x_1(t_0)\| \le \|x(t_0)\|$, $\|x_2(t_0)\| \le \|x(t_0)\|$ then

$$\|x(t)\| \le \beta(\|x(t_0)\|, t-t_0),$$

where

$$\beta(r,s) = \beta_1(\beta_1(r, s/2) + \gamma_1(\beta_2(r,0)), s/2) + \gamma_1(\beta_2(r, s/2)) + \beta_2(r,s)$$

ISS Small-Gain Theorem

Consider the interconnected system

$$\dot{x}_1 = f_1(t, x_1, x_2, w_1)$$
$$\dot{x}_2 = f_2(t, x_1, x_2, w_2)$$

Assume that $f_1(t, 0, 0, 0) = f_2(t, 0, 0, 0, 0) = 0, \forall t \geq t_0 \geq 0$ where $f_1(t, ., ., .), f_2(t, ., ., .)$ are piecewise continuous in t and locally Lipschitz in the rest of arguments. Suppose that

- x_1 subsystem is ISS with respect to state x_1 and inputs (x_2, w_1), that is $\exists_{\beta_1 \in \mathcal{KL}} \exists_{\gamma_1, \gamma_{w_1} \in \mathcal{K}}$ such that for all $x_1(t_0)$ and any bounded (x_2, w_1)

$$\|x_1(t)\| \leq \beta_1(\|x_1(t_0)\|, t - t_0) + \gamma_1(\sup_{\tau \in [t_0, t]} \|x_2(\tau)\|) + \gamma_{w_1}(\sup_{\tau \in [t_0, t]} \|w_1(\tau)\|)$$

- x_2 subsystem is ISS with respect to state x_2 and inputs (x_1, w_2)
- $\gamma_1 \circ \gamma_2(r) < r \; \forall r > 0$ (contraction condition)

Then, the interconnect system is ISS with respect to state $x = [x_1 \;\, x_2]^T$ and input $w = [w_1 \;\, w_2]^T$