Nonlinear Control Systems

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2. Mathematical review

IST-DEEC PhD Course

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Mathematical review (Euclidean Space)

Vector and matrix norms

The norm $\|x\|$ of a vector $x \in \mathbb{R}^n$ is a real valued function with the properties

1. $\|x\|$, $\forall x \in \mathbb{R}^n$, with $\|x\| = 0$ if and only if $x = 0$,
2. $\|x + y\| \leq \|x\| + \|y\|$, $\forall x, y \in \mathbb{R}^n$ (Triangular inequality)
3. $\|\alpha x\| = |\alpha|\|x\|$, $\forall \alpha \in \mathbb{R}$ and $\forall x \in \mathbb{R}^n$.

p-norm

$$\|x\|_p = (|x_1|^p + |x_2|^p + \ldots + |x_n|^p)^{1/p}, \qquad 1 \leq p < \infty$$

∞-norm

$$\|x\|_\infty = \max_i |x_i|$$

The three most commonly used norms are $\|x\|_1$, $\|x\|_\infty$ and the Euclidean norm

$$\|x\|_2 = \left(|x_1|^2 + |x_2|^2 + \ldots + |x_n|^2\right)^{1/2}$$

Mathematical review (Euclidean Space)

Properties

- For $\|.\|_\alpha$,$\|.\|_\beta$, with $\alpha \neq \beta$ there exist positive constants c_1, c_2 such that $c_1 \|x\|_\alpha \leq \|x\|_\beta \leq c_2 \|x\|_\alpha$, $\forall x \in \mathbb{R}^n$. In particular
  - $\|x\|_2 \leq \|x\|_1 \leq \sqrt{n} \|x\|_2$
  - $\|x\|_\infty \leq \|x\|_2 \leq \sqrt{n} \|x\|_\infty$
  - $\|x\|_\infty \leq \|x\|_1 \leq n \|x\|_\infty$
- Hölder inequality

$$\left|x^T y\right| \leq \|x\|_p \|y\|_q, \quad \frac{1}{p} + \frac{1}{q} = 1, \forall x, y \in \mathbb{R}^n$$

Mathematical review (Euclidean Space)

Matrix norms The induced p-norm of $A \in \mathbb{R}^{m \times n}$

$$\|A\|_p = \sup_{x \neq 0} \frac{\|Ax\|_p}{\|x\|_p} = \sup_{\|x\|_p = 1} \|Ax\|_p$$

In particular

- $p = 1$

$$\|A\|_1 = \max_j \sum_{i=1}^{m} |a_{ij}|$$

  (The maximum by columns)

- $p = 2$

$$\|A\|_2 = \lambda_{\max}\left(A^T A\right)^{1/2}$$

  where $\lambda_{\max}(A^T A)$ is the maximum eigenvalue of $A^T A$.

- $p = \infty$

$$\|A\|_\infty = \max_i \sum_{j=1}^{n} |a_{ij}|$$

  (The maximum by rows)

Mathematical review (Euclidean Space)

Some useful properties. Let $A \in \mathbb{R}^{m \times n}$ and $B \in \mathbb{R}^{n \times l}$

1. $$\frac{1}{\sqrt{n}}\|A\|_\infty \leq \|A\|_2 \leq \sqrt{m}\|A\|_\infty$$
2. $$\frac{1}{\sqrt{m}}\|A\|_1 \leq \|A\|_2 \leq \sqrt{n}\|A\|_1$$
3. $$\|A\|_2 \leq \sqrt{\|A\|_1\|A\|_\infty}$$
4. $$\|AB\|_p \leq \|A\|_p\|B\|_p$$

Topological concepts in $\mathbb{R}^n$

Convergence of sequences

- A sequence of vectors $x_0, x_1, ..., x_k, ... \in \mathbb{R}^n$ denoted by $\{x_k\}$, is said to converge to a limit vector x if

$$\|x_k - x\| \to 0$$

  as $k \to \infty$, which is equivalent to

$$\forall_{\varepsilon > 0} \, \exists_N : \|x_k - x\| < \varepsilon, \ \forall_{k \geq N}$$

- A vector x is an *accumulation point* of a sequence $\{x_k\}$ if there is a subsequence of $\{x_k\}$ that converges to x. That is, if there is an infinite subset K of the nonnegative integers such that $\{x_k\}_{k \in K}$ converges to x.
- A bounded sequence $\{x_k\} \in \mathbb{R}^n$ has at least one accumulation point in $\mathbb{R}^n$.
- Increasing sequence: $\{x_k\} \in \mathbb{R}^n$, $x_k \leq x_{k+1}, \ \forall k$. If $\{x_k\} \in \mathbb{R}^n$, $x_k < x_{k+1}, \ \forall k$ it is said to be *strictly increasing*.
- Decreasing sequence $\{x_k\} \in \mathbb{R}^n$, $x_k \geq x_{k+1}, \ \forall k$. If $\{x_k\} \in \mathbb{R}^n$, $x_k > x_{k+1}, \ \forall k$ it is said to be *strictly decreasing*.
- An increasing sequence $\{x_k\} \in \mathbb{R}^n$ that is bounded from above converges to a real number.
- Similarly, a decreasing sequence that is bounded from below converges to a real number.

Topological concepts in $\mathbb{R}^n$

Sets Let $S \subset \mathbb{R}^n$

- Open set: $\forall x \in S$, one can find an ϵ-neighborhood of x

$$N(x, \epsilon) = \{z \in \mathbb{R}^n : \|z - x\| < \epsilon\}$$

  such that $N(x, \epsilon) \subset S$

- Closed set: Every convergent sequence $\{x_k\}$ with elements in S converges to a point in S.
- Bounded set: S is bounded if there is $r > 0$ such that $\|x\| < r, \ \forall x \in S$
- Compact set: If it is closed and bounded.
- Boundary point: A point p is a boundary point of a set S if every neighborhood of p contains at least one point of S and one point not belonging to S.

The set of all boundary points of S is denoted by ∂S, the interior of a set by $S - \partial S$ and the closure of a set $\bar{S} = S \cup \partial S$.

Mathematical review (Euclidean Space)

Continuous functions

A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be continuous at a point x if $f(x_k) \to f(x)$ whenever $x_k \to x$. Equivalently, if

$$\forall_{\varepsilon>0}\ \exists_{\delta>0} : \|x-y\| < \delta \Rightarrow \|f(x) - f(y)\| < \varepsilon$$

Differentiable functions

A function $f : \mathbb{R} \to \mathbb{R}$ is said to be differentiable at x if the limit

$$f'(x) = \lim_{h \to 0} \frac{f(x+h) - f(x)}{h}$$

exists. A function $f : \mathbb{R}^n \to \mathbb{R}^m$ is said to be *continuously differentiable* (denoted as $f \in C^1$) at a point x_0 if the partial derivatives $\frac{\partial f_i}{\partial x_j}$ exist and are continuous at x_0 for $1 \leq i \leq m,\ 1 \leq j \leq n$.

Mathematical review (Euclidean Space)

Gradient vector

For a continuously differentiable function $f : \mathbb{R}^n \to \mathbb{R}$

$$\nabla f(x) = \begin{bmatrix} \frac{\partial f}{\partial x_1} & \frac{\partial f}{\partial x_2} & \cdots & \frac{\partial f}{\partial x_n} \end{bmatrix}^T$$

Jacobian Matrix

For a continuous differentiable function $f : \mathbb{R}^n \to \mathbb{R}^m$

$$J(x) = \left[\frac{\partial f}{\partial x}\right] = \begin{bmatrix} \frac{\partial f_1}{\partial x_1} & \cdots & \frac{\partial f_1}{\partial x_n} \\ \vdots & \ddots & \vdots \\ \frac{\partial f_m}{\partial x_1} & \cdots & \frac{\partial f_m}{\partial x_n} \end{bmatrix}$$

Chain rule

Let $h(x) = g(f(x))$ be a continuously differentiable function at x_0. Then the Jacobian matrix is given by the chain rule

$$\left.\frac{\partial h}{\partial x}\right|_{x=x_0} = \left.\frac{\partial g}{\partial f}\right|_{f=f(x_0)} \left.\frac{\partial f}{\partial x}\right|_{x=x_0}$$

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Mean value theorem

Let $f:\mathbb{R}^n \to \mathbb{R}$ be a continuously differentiable function at each point x of an open set $S \subset \mathbb{R}^n$. Let x and y be two points of S such that the line segment $L(x,y) = \{z : z = \theta x + (1-\theta)y,\ 0 \le \theta \le 1\}$ belongs to S. Then exists a point z of $L(x,y)$ such that

$$f(y) - f(x) = \left.\frac{\partial f}{\partial x}\right|_{x=z} (y-x)$$

Implicit Function Theorem

Let $f:\mathbb{R}^n \to \mathbb{R}^m$ be continuously differentiable at each point (x,y) of an open set $S \subset \mathbb{R}^n \times \mathbb{R}^m$. Let $(x_0,y_0) \in S$ be a point such that $f(x_0,y_0) = 0$ and for which the Jacobian matrix $\frac{\partial f}{\partial x}(x_0,y_0)$ is nonsingular. Then there exist neighborhoods $U \subset \mathbb{R}^n$ of x_0 and $V \subset \mathbb{R}^m$ of y_0 such that for each $y \in V$ the equation $f(x,y) = 0$ has a unique solution $x \in U$. Moreover, this solution can be given as $x = g(y)$, where g is continuously differentiable at $y = y_0$.

Mathematical review

Grownwall-Bellman Inequality

Let $\lambda : [a,b] \to \mathbb{R}$ be continuous and $\mu : [a,b] \to \mathbb{R}$ be continuous and nonnegative. If a continuous function $y : [a,b] \to \mathbb{R}$ satisfies

$$y(t) \le \lambda(t) + \int_a^t \mu(s)\, y(s)\, ds, \qquad a \le t \le b$$

then

$$y(t) \le \lambda(t) + \int_a^t \lambda(s)\, \mu(s)\, e^{\int_s^t \mu(\tau) d\tau} ds$$

In particular, if $\lambda(t) = \lambda$ is a constant, then

$$y(t) \le \lambda\, e^{\int_a^t \mu(\tau) d\tau}$$

If in addition, $\mu(t) = \mu > 0$ is a constant, then

$$y(t) \le \lambda\, e^{\mu(t-a)}$$