Nonlinear Control Systems

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1. Introduction to Nonlinear Systems

IST-DEEC PhD Course

http://users.isr.ist.utl.pt/%7Epedro/NCS2012/

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Objective The main goal of this course is to provide to the students a solid background in analysis and design of nonlinear control systems

Why analysis? (and not only simulation)

- Every day computers are becoming more and more powerful to simulate complex systems
- Simulation combined with good intuition can provide useful insight into system's behavior

Nevertheless

- It is not feasible to rely only on simulations when trying to obtain *guarantees* of stability and performance of nonlinear systems, since crucial cases may be missed
- Analysis tools provide the means to obtain formal mathematical proofs (certificates) about the system's behavior
- results may be surprising, i.e, something we had not thought to simulate.

Why study nonlinear systems?

Nonlinear versus linear systems

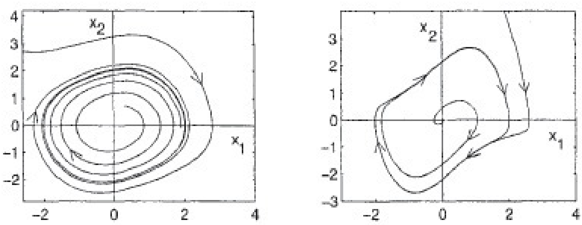
- Huge body of work in analysis and control of linear systems
- most models currently available are linear (but most real systems are nonlinear...)

However

- dynamics of linear systems are not rich enough to describe many commonly observed phenomena

Examples of essentially nonlinear phenomena

- Finite escape time, i.e, the state can go to infinity in finite time (while this is impossible to happen for linear systems)
- Multiple isolated equilibria, while linear systems can only have one isolated equilibrium point, that is, one steady state operating point
- Limit cycles (oscillation of fixed amplitude and frequency, irrespective of the initial state)



- Subharmonic, harmonic or almost-periodic oscillations;
  - A stable linear system under a periodic input produces an output of the same frequency.
  - A nonlinear system can oscillate with frequencies which are submultiples or multiples of the input frequency. It may even generate an almost-periodic oscillation, i.e, sum of periodic oscillations with frequencies which are not multiples of each other.
- Other complex dynamic behavior, for example: chaos, biforcations, discontinuous jump, etc...

State-space model

State equation

$$\dot{x} = f(t, x, u)$$

Output equation

$$y = h(t, x, u)$$

$$x = \begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}, \quad u = \begin{bmatrix} u_1 \\ u_2 \\ \vdots \\ u_m \end{bmatrix}, \quad f(t, x, u) = \begin{bmatrix} f_1(t, x, u) \\ f_2(t, x, u) \\ \vdots \\ f_n(t, x, u) \end{bmatrix}$$

where $x \in \mathbb{R}^n$ is the state variable, $u \in \mathbb{R}^m$ is the input signal, and $y \in \mathbb{R}^q$ the output signal. The symbol $\dot{x} = \frac{dx}{dt}$ denotes the derivative of x with respect to time t.

State-space model

State equation

$$\dot{x} = f(t, x, u) \tag{1}$$

Output equation

$$y = h(t, x, u) \tag{2}$$

Particular cases:

- Linear Systems, where the state model takes the form

$$\dot{x} = A(t)x + B(t)u$$

$$y = C(t)x + D(t)u$$

- Unforced state equation

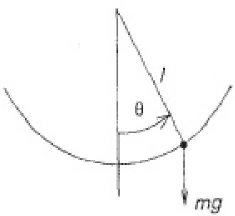
$$\dot{x} = f(t, x)$$

  i.e., it does not depend explicitly on the input u, e.g., consider the case that there is a state feedback $u = \gamma(t, x)$, and therefore the closed-loop system is given by $\dot{x} = f(t, x, \gamma(t, x)) = \tilde{f}(t, x)$

- Unforced autonomous (or time-invariant) state equation

$$\dot{x} = f(x)$$

Example - Pendulum

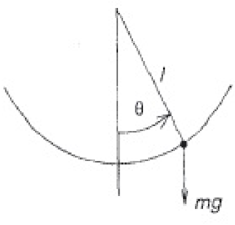


There is a frictional force assumed to be proportional to the (linear) speed of the mass m. Using the Newton's second law of motion at the tangential direction

$$ml\ddot{\theta} = -mg\sin(\theta) - kl\dot{\theta}$$

where m is the mass, l is the length of the rope and k the frictional constant.

Example - Pendulum



$$ml\ddot{\theta} = -mg\sin(\theta) - kl\dot{\theta}$$

State model

$$x_1 = \theta$$
$$\dot{x}_1 = \dot{\theta}$$

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2$$

What are the equilibrium points?

Equilibrium point

A point $x = x^*$ in the state space is said to be an equilibrium point of

$$\dot{x} = f(t, x)$$

if

$$x(t_0) = x^* \quad \Rightarrow \quad x(t) = x^*, \forall t \geq t_0$$

that is, if the state starts at x^*, it will remain at x^* for all future time.

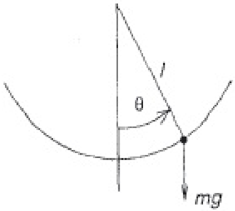
For autonomous systems, the equilibrium points are the real roots of $f(x) = 0$.

The equilibrium points can be of two kinds:

- isolated, that is, there are no other equilibrium points in its vicinity
- continuum of equilibrium points.

Much of nonlinear analysis is based on studying the behavior of a system around its equilibrium points.

Example - Pendulum



State model

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2$$

Equilibrium points:

$$0 = x_2$$
$$0 = -\frac{g}{l}\sin(x_1) - \frac{k}{m}x_2$$

which implies that $x_2 = 0$ and $\sin(x_1) = 0$. Thus, the equilibrium points are

$$(n\pi, 0), \ n \in \mathbb{Z}$$

What is the behavior of the system near the equilibrium points?

Qualitative behavior of 2nd order linear time-invariant systems

$$\dot{x} = Ax, \; x \in \mathbb{R}^2, A \in \mathbb{R}^{2\times 2}$$

Apply a similarity transformation M to A:

$$M^{-1}AM = J, \; M \in \mathbb{R}^{2\times 2}$$

where J is the real *Jordan form* of A, which depending on the eigenvalues of A may take one of the three forms

$$\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}, \quad \begin{bmatrix} \lambda & k \\ 0 & \lambda \end{bmatrix}, \quad \begin{bmatrix} \alpha & -\beta \\ \beta & \alpha \end{bmatrix}$$

with k being either 0 or 1.

Case 1: Both eigenvalues are real with $\lambda_1 \neq \lambda_2 \neq 0$

$$J = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix}$$

What is M?
The associated eigenvectors $v_1, v_2 \in \mathbb{R}^{2\times 1}$ must satisfy

$$\begin{aligned} Av_1 &= \lambda_1 v_1 \\ Av_2 &= \lambda_2 v_2 \end{aligned}$$

Thus,

$$A\,[v_1|v_2] = [v_1|v_2]\begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} \quad \Rightarrow \quad M = [v_1|v_2] \quad \Rightarrow \quad M^{-1}AM = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} = J$$

This represents a change of coordinates

$$z = M^{-1}x$$

and we obtain in the new referential

$$\begin{aligned} \dot{z}_1 &= \lambda_1 z_1 \\ \dot{z}_2 &= \lambda_2 z_2 \end{aligned}$$

Why?

$$\dot{z} = M^{-1}\dot{x} = M^{-1}Ax = M^{-1}AMz = \begin{bmatrix} \lambda_1 & 0 \\ 0 & \lambda_2 \end{bmatrix} z.$$

Case 1: Both eigenvalues are real with $\lambda_1 \neq \lambda_2 \neq 0$

$$\begin{aligned}\dot{z}_1 &= \lambda_1 z_1 \\ \dot{z}_2 &= \lambda_2 z_2\end{aligned}$$

For a given initial state $(z_1, z_2)(0)$, the solution is given by

$$\begin{aligned}z_1(t) &= z_1(0)e^{\lambda_1 t} \\ z_2(t) &= z_2(0)e^{\lambda_2 t}\end{aligned}$$

Eliminating time t,

$$z_2(t) = \frac{z_2(0)}{z_1(0)^{\lambda_2/\lambda_1}} z_1(t)^{\lambda_2/\lambda_1}$$

Why?

$$\frac{z_1(t)}{z_1(0)} = e^{\lambda_1 t} \rightarrow t = \frac{1}{\lambda_1} \ln \frac{z_1(t)}{z_1(0)}$$

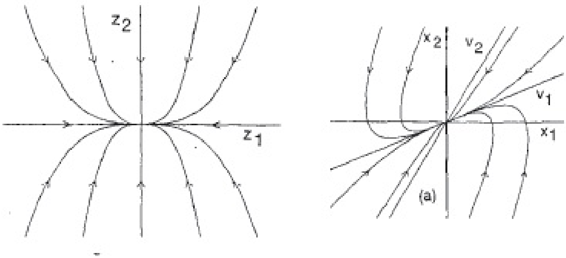
and so,

$$z_2(t) = z_2(0)e^{\frac{\lambda_2}{\lambda_1} \ln \frac{z_1(t)}{z_1(0)}} = z_2(0)e^{\ln \frac{z_1(t)}{z_1(0)}^{\frac{\lambda_2}{\lambda_1}}}.$$

At this point several combinations of the eigenvalues can arise...

Case 1: Both eigenvalues are real with $\lambda_1 \neq \lambda_2 \neq 0$

(a) $\lambda_1, \lambda_2 < 0$. In this case $e^{\lambda_1 t}, e^{\lambda_1 t} \to 0$ and the curves are parabolic. Consider, without loss of generality that $\lambda_2 < \lambda_1$.
Phase portrait



The equilibrium point $x = 0$ is called a *stable node*.

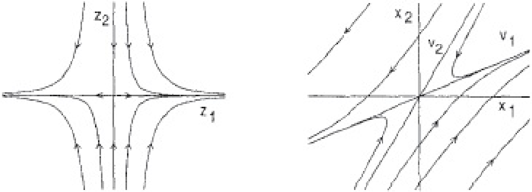
Case 1: Both eigenvalues are real with $\lambda_1 \neq \lambda_2 \neq 0$

(b) $\lambda_1, \lambda_2 > 0$. The phase portrait will retain the same character but with the trajectories directions reversed. In this case the equilibrium point $x = 0$ is called an *unstable node*.

(c) The eigenvalues have opposite signs. Consider for example the case $\lambda_2 < 0 < \lambda_1$.

$$z_2(t) = z_2(0) e^{\ln \frac{z_1(t)}{z_1(0)}^{\frac{\lambda_2}{\lambda_1}}}.$$

The exponent $\frac{\lambda_2}{\lambda_1}$ is negative, thus we have hyperbolic curves.



In this case, the equilibrium point is called a *saddle point*.

Case 2: Complex eigenvalues $\lambda_{1,2} = \alpha \pm j\beta$, $\alpha, \beta \in \mathbb{R}$

$$J = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

Associated eigenvectors

$$\begin{aligned} v_1 &= u + jv \\ v_2 &= u - jv, \quad u, v \in \mathbb{R}^2 \end{aligned}$$

What is M?

$$\begin{aligned} A(u + jv) &= (\alpha + j\beta)(u + jv) \\ A(u - jv) &= (\alpha - j\beta)(u - jv) \end{aligned}$$

Thus,

real part $\rightarrow$ $Au = \alpha u - \beta v$

imaginary part $\rightarrow$ $Av = \beta u + \alpha v$

Rearranging we have

$$A[u|v] = [u|v] \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix} = J$$

which implies that

$$M = [u|v] \rightarrow M^{-1}AM = \begin{bmatrix} \alpha & \beta \\ -\beta & \alpha \end{bmatrix}$$

Coordinate transformation

$z = M^{-1}x$, then

$$\begin{aligned} \dot{z}_1 &= \alpha z_1 - \beta z_2, \\ \dot{z}_2 &= \beta z_1 + \alpha z_2 \end{aligned}$$

Case 2: Complex eigenvalues $\lambda_{1,2} = \alpha \pm j\beta$, $\alpha, \beta \in \mathbb{R}$

Better insight into the solution if we work in polar coordinates,

$$r = \sqrt{z_1^2 + z_2^2},$$
$$\theta = \tan^{-1}\left(\frac{z_2}{z_1}\right).$$

where in this case we have the following system

$$\dot{r} = \alpha r$$
$$\dot{\theta} = \beta$$

The solution is given by

$$r(t) = r(0)e^{\alpha t}$$
$$\theta(t) = \theta(0) + \beta t$$

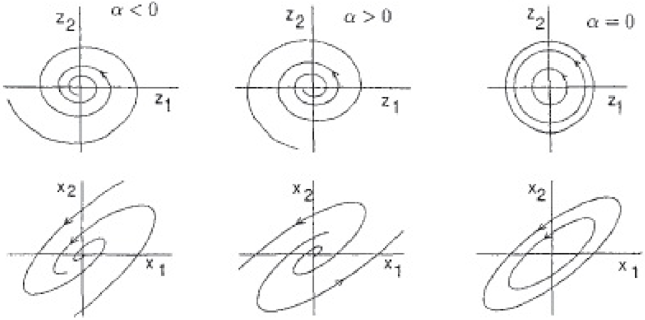
Case 2: Complex eigenvalues $\lambda_{1,2} = \alpha \pm j\beta$, $\alpha, \beta \in \mathbb{R}$

These equations represent the logarithmic spiral, and for different values of α we get

i) $\alpha < 0$ is a stable focus

ii) $\alpha > 0$ is an unstable focus

iii) $\alpha = 0$ is a center.



stable focus; unstable focus a center

Other cases

a) $\lambda_1 = \lambda_2 = \lambda \neq 0$

$$\dot{z}_1 = \lambda z_1 + k z_2, \quad k = 0 \text{ or } k = 1$$
$$\dot{z}_2 = \lambda z_2$$

b) $\lambda_1 = 0, \lambda_2 \neq 0$

$$\dot{z}_1 = 0$$
$$\dot{z}_2 = \lambda z_2$$

c) $\lambda_1 = \lambda_2 = 0$

$$\dot{z}_1 = z_2, \quad (k = 1)$$
$$\dot{z}_2 = 0$$

⋮

Extensions to nonlinear systems (2nd order)

Consider the 2nd order nonlinear time invariant system

$$\dot{x}_1 = f_1(x_1, x_2)$$
$$\dot{x}_2 = f_2(x_1, x_2)$$

where $f = (f_1, f_2)^T$ is continuously differentiable. Moreover assume that

$$x^* = \begin{bmatrix} x_1^* \\ x_2^* \end{bmatrix}$$

is an isolated equilibrium point, i.e,

$$0 = f_1(x_1^*, x_2^*)$$
$$0 = f_2(x_1^*, x_2^*)$$

Expanding the right-hand side into its Taylor series around x^*

$$\dot{x}_1 = f_1(x_1^*, x_2^*) + a_{11}(x_1 - x_1^*) + a_{12}(x_2 - x_2^*) + H.O.T.$$
$$\dot{x}_2 = f_2(x_1^*, x_2^*) + a_{21}(x_1 - x_1^*) + a_{22}(x_2 - x_2^*) + H.O.T.$$

where

$$a_{11} = \left.\frac{\partial f_1(x_1^*, x_2^*)}{\partial x_1}\right|_{x_1 = x_1^*, x_2 = x_2^*} \qquad a_{12} = \left.\frac{\partial f_1(x_1^*, x_2^*)}{\partial x_2}\right|_{x_1 = x_1^*, x_2 = x_2^*}$$
$$a_{21} = \left.\frac{\partial f_2(x_1^*, x_2^*)}{\partial x_1}\right|_{x_1 = x_1^*, x_2 = x_2^*} \qquad a_{22} = \left.\frac{\partial f_2(x_1^*, x_2^*)}{\partial x_2}\right|_{x_1 = x_1^*, x_2 = x_2^*}$$

Extensions to nonlinear systems (2nd order)

Defining $z = [z_1, z_2]^T$, $z_i = x_i - x_i^*,\ i = 1, 2$,

$$A = \begin{bmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{bmatrix}$$

also denoted as the Jacobian Matrix and considering only the first order terms we obtain the following linear system

$$\dot{z} = Az.$$

The question now is...

What can we conclude about the behavior of the nonlinear system around an equilibrium point from the study of the linearized system?

Extensions to nonlinear systems (2nd order)

Consider the 2nd order nonlinear time invariant system

$$\begin{aligned} \dot{x}_1 &= f_1(x_1, x_2) \\ \dot{x}_2 &= f_2(x_1, x_2) \end{aligned} \qquad \Leftrightarrow \qquad \dot{x} = f(x)$$

where f_1, f_2 are analytic (i.e., f_1, f_2 have convergent Taylor series representation) and $f(x^*) = 0$. The linearization around the equilibrium point $x = x^*$ provides the following linear system

$$\dot{z} = Az, \qquad A = \left.\frac{\partial f}{\partial x}\right|_{x=x^*}$$

where $z = x - x^*$.

- If the origin $z = 0$ of the linearized state equation is a *stable (resp. unstable) node, or a stable (resp. unstable) focus or a saddle point*, then in a small neighborhood of the equilibrium point, the trajectories of the nonlinear system will behave like a *stable (resp. unstable) node, or a stable (resp, unstable) focus or a saddle point*, respectively.

Extensions to nonlinear systems (2nd order)

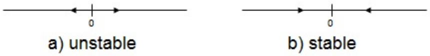
However

If the Jacobian matrix A has eigenvalues on the imaginary axis, then the qualitative behavior of the nonlinear state equation near the equilibrium point could be quite distinct from that of the linearized state equation!

Example:

a) $\dot{x} = x^3, \; x \in \mathbb{R}$, is an unstable system

b) $\dot{x} = -x^3, \; x \in \mathbb{R}$ is a stable system



But they have the same linearization! (i.e., $\dot{z} = 0$)

Example - Pendulum

State model

$$\dot{x}_1 = x_2$$
$$\dot{x}_2 = -\tfrac{g}{l}\sin(x_1) - \tfrac{k}{m}x_2$$

Equilibrium points

$$(n\pi, 0),\ n \in \mathbb{Z}$$

Jacobian matrix

$$\left.\frac{\partial f}{\partial x}\right|_{x=x^*} = \left.\begin{bmatrix} \frac{\partial f_1(x_1,x_2)}{\partial x_1} & \frac{\partial f_1(x_1,x_2)}{\partial x_2} \\ \frac{\partial f_2(x_1,x_2)}{\partial x_1} & \frac{\partial f_2(x_1,x_2)}{\partial x_2} \end{bmatrix}\right|_{x_1=x_1^*,x_2=x_2^*} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l}\cos(x_1^*) & -\frac{k}{m} \end{bmatrix}$$

1. $x^* = (0,0)$

$$A_1 = \left.\frac{\partial f}{\partial x}\right|_{x_1=0,x_2=0} = \begin{bmatrix} 0 & 1 \\ -\frac{g}{l} & -\frac{k}{m} \end{bmatrix} \rightarrow \lambda_{1,2} = -\frac{k}{2m} \pm \frac{1}{2}\sqrt{\left(\frac{k}{m}\right)^2 - \frac{4g}{l}}$$

2. $x^* = (\pi,0)$

$$A_2 = \left.\frac{\partial f}{\partial x}\right|_{x_1=\pi,x_2=0} = \begin{bmatrix} 0 & 1 \\ \frac{g}{l} & -\frac{k}{m} \end{bmatrix} \rightarrow \lambda_{1,2} = -\frac{k}{2m} \pm \frac{1}{2}\sqrt{\left(\frac{k}{m}\right)^2 + \frac{4g}{l}}$$

Example - Pendulum

Consider the case $\frac{g}{l} = 1$ and $\frac{k}{m} = 0.5$.

1. $z_1 = x - 0$

$$\dot{z}_1 = A_1 z_1, \quad A_1 = \begin{bmatrix} 0 & 1 \\ -1 & -0.5 \end{bmatrix}$$

   Eigenvalues: $\lambda_{1,2} = -0.25 \pm j0.97$. Thus the equilibrium point $x^* = (0,0)$ is a stable focus.

2. $z_2 = x - \pi$

$$\dot{z}_2 = A_2 z_2, \quad A_2 = \begin{bmatrix} 0 & 1 \\ 1 & -0.5 \end{bmatrix}$$

   Eigenvalues: $\lambda_1 = -1.28,\ \lambda_2 = 0.78$. Thus the equilibrium point $x^* = (\pi, 0)$ is a saddle point.

If $k = 0$, $\lambda_{1,2}$ are on the imaginary axis and therefore we cannot determine the stability of the origin through linearization!

Example 3

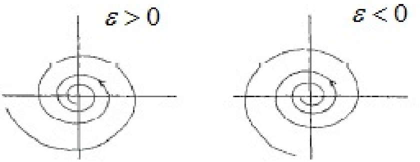
Consider the following system

$$\dot{x}_1 = x_2,$$
$$\dot{x}_2 = -x_1 - \varepsilon x_1^2 x_2$$

The linearization around $x = 0$ yields

$$\begin{bmatrix} \dot{z}_1 \\ \dot{z}_2 \end{bmatrix} = \begin{bmatrix} 0 & 1 \\ -1 & 0 \end{bmatrix} \begin{bmatrix} z_1 \\ z_2 \end{bmatrix}$$

where the eigenvalues are $\pm j$.
Is it a center?
The answer is NO, for $\epsilon > 0$ is stable and for $\epsilon < 0$ is unstable.



stable   unstable

How do we conclude its stability?
Tip: check the evolution of the "energy"

$$\frac{d}{dt}\left(x_1^2 + x_2^2\right) = -2\varepsilon x_1^2 x_2^2$$

Example 4

Consider the following system

$$\dot{x}_1 = \mu - x_1^2$$
$$\dot{x}_2 = -x_2^2$$

where $\mu \in \mathbb{R}$ is a small parameter.

a) For $\mu > 0$ there exist two equilibrium points at $(\sqrt{\mu}, 0)$ and $(-\sqrt{\mu}, 0)$. Performing the linearization we get

$$A = \begin{bmatrix} -2\sqrt{\mu} & 0 \\ 0 & -1 \end{bmatrix}$$

for $(\sqrt{\mu}, 0)$, which is a stable node and

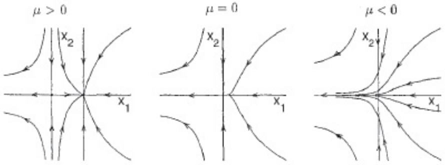
$$A = \begin{bmatrix} 2\sqrt{\mu} & 0 \\ 0 & -1 \end{bmatrix}$$

for $(-\sqrt{\mu}, 0)$, which is a saddle point.

b) For $\mu < 0$ there are no equilibrium points.

Example 4

Phase portraits:



- We are in the presence of a *bifurcation*, that is, a change in the equilibrium points or periodic orbits or in their stability properties, as a parameter is varied.
- In this example we have a saddle-node bifurcation because it results from the collision of a saddle and a node. μ is the bifurcation parameter and $\mu = 0$ is the bifurcation point.
- There exist other types of bifurcations, e.g. transcritical bifurcation, super/subcritical pitchfork bifurcation and super/subcritical Hopf bifurcation.