A FAST MAP ALGORITHM USING HIGH ORDER GIBBS PRIORS

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ABSTRACT

The estimation algorithms based on the MAP (Maximum-a-Posteriori) criterion have the ability to deal with ill-posed problems, involving noisy data, by reducing the number of admissible solutions. However, this method is usually iterative and time consuming and not appropriate to work in a real time basis. In the context of image reconstruction/restoration, Gibbs priors with quadratic potential functions have been used because they allow simple mathematical formulations. However, these priors lead to smooth solutions with a poor representation of transitions. To overcome this problem, several authors have proposed other potential function, non quadratic, that deal better with the transitions, but leading to an increase of complexity.

In this paper, we come back to the Gibbs priors with quadratic potential functions to show that MAP algorithms can be formulated as a linear filtering problem. The recursive nature of the filtering methods allows the obtention of very fast and efficient restoration/reconstruction MAP algorithms. Furthermore, we show that by using quadratic potential functions involving high order differences between neighbors it is possible to significantly improve the solution at the transitions. In fact, with this strategy we are increasing the order of the filter that models the original MAP problem.

The approach presented in this paper can be used as a method to design Gibbs priors based on the filter design theory.

1. INTRODUCTION

The estimation of signals from noisy observations has been thoroughly studied in the last three decades. MAP methods using MRF (Markov Random Fields) models are powerful tools to address this problem, taking into account the statistics of the observations as well as the prior information available about the signal to estimate. Gibbs priors [1, 2] have been extensively used for this purpose. They are based on the choice of a potential function which penalizes abrupt changes of the signal.

A popular solution consists of using quadratic potential functions based on first order differences [3], leading to simple estimation algorithms which efficiently reduce the observation noise. Unfortunately, quadratic potentials tend to smooth the transitions and to eliminate small details of the signal. Noise reduction is achieved by smoothing the signal, destroying high frequencies. To overcome this difficulty several strategies have been followed, namely by the use of non-quadratic potential functions [4, 5] and multi-scale representations of the signal (e.g., wavelets)[6]. These methods lead to an improvement at the transitions but they rely on iterative optimization schemes with higher computational complexity. Furthermore the convergence can be slow.

In previous publications [7], the authors have shown that the MAP solution with first order Gibbs priors can be obtained by a linear filtering operation over the maximum likelihood solution. Here, the authors have derived the coefficients of the filter, as function of the prior parameters.

In this paper we describe the reverse operation, i.e., we derive the prior parameters from the coefficients of the filter which is a much more simple and useful operation. In fact, using this method, it is possible to use the filtering design theory to design quadratic priors with arbitrary shape in the frequency domain. By using higher order (> 1) quadratic potential functions and a non interactive optimization procedure based on IIR (Infinite Impulse Response) filtering it is possible to obtain fast and low computational complexity algorithms with good performance at transitions.

2. PROBLEM FORMULATION

Let $F = \{f_i\}$ be a vector of N unknowns to be estimated and $Y = \{y_i\}$ a vector of N noisy observations of F. The observations are assumed to be statistically independent following a normal distribution. Therefore, the log likelihood function is

$$l(Y,F) = \sum_{i=1}^{N} \log(p(y_i)) = -\frac{1}{2\sigma^2}\sum_{i=1}^{N}(y_i - f_i)^2 + C \quad (1)$$

In this paper the vector F follows a Gibbs distribution

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with quadratic potential functions of order v, i.e.,

$$p(F) = \frac{1}{Z} e^{-\frac{\psi}{2B} \sum_{i=1}^{N} \sum_{j=1}^{v} \beta_j (f_i - f_{i-j})^2} \tag{2}$$

where $B = \sum_{j=1}^{v} \beta_j$ is a normalizing factor and $\beta_1 = 1$

2.1. MAP solution

The MAP solution is obtained by solving

$$\hat{F} = arg \min_F E(Y, F) \tag{3}$$

where

$$E(Y, F) = -l(Y, F) - \log p(F) = \tag{4}$$

$$\frac{1}{2\sigma^2} \sum_{i=1}^{N} (y_i - f_i)^2 + \frac{\psi}{2B} \sum_{i=1}^{N} \sum_{j=1}^{v} \beta_j (f_i - f_{i-j})^2 + C \tag{5}$$

Using the Gauss-Seidel algorithm and the fixed point method to find the stationary point of $E(Y, F)$ with respect to each f_i, we obtain the following recursion

$$f_i^{n+1} = (1 - k_i) f_i^{ML} + k_i \bar{f}_i^n \tag{6}$$

$$k_i = k = \frac{1}{1 + 1/(2\psi\sigma^2)} \tag{7}$$

$$\bar{f}_i = \frac{1}{2B} \sum_{j=1}^{v} \beta_j (f_{i-j} + f_{i+j}) \tag{8}$$

where $f_i^{ML} = y_i$. In this denoising problem, we only have an observation per unknown. Therefore, the observations are equal to the maximum likelihood estimates. All the coefficients are updated according to (6)-(8). The estimation procedure is repeated until convergence is achieved.

2.2. IIR filter

The potential functions in (2) consider not only the closest neighbors but other neighbors as well, to improve the solution at the transitions.

However, choosing the prior coefficients in a trial and error basis is not pratical, specially when v is large.

This section shows that the MAP solution can be obtain in two steps, by filtering the maximum likelihood estimates with a causal and an anti-causal filter. We also show how to derive the parameters ψ and β_j of the prior from the filter coefficients. Therefore, the selection of the parameters β_j is converted into a filter design problem.

Equation (6) can be seen as a filtering process. By computing its Z transform we obtain

$$G(z) = \frac{F_z}{Y_z} = \frac{1 - k}{1 - \frac{k}{2B} \sum_{j=1}^{v} \beta_j (z^{-j} + z^j)} \tag{9}$$

where F_z and Y_z are the Z transform of F and Y respectively.

This filter can not be recursively computed because it is not wedge supported, i.e., each output depends on past and future outputs. To overcome this difficulty it is decomposed as a cascade of two filters, a causal and an anti-causal filter

$$C(z) = H_c(z) H_a(z) \tag{10}$$

whith

$$H_c(z) = \frac{A}{\sum_{j=0}^{v} a_j z^{-j}} \qquad H_a(z) = \frac{A}{\sum_{j=0}^{v} a_j z^j} \tag{11}$$

where $A = \sum_{j=0}^{v} a_j$. These filters have unit gain at DC, i.e., $H_c(1) = H_a(1) = 1$ and zeros only at the origin. This leads to

$$H_c(z) H_a(z) = \frac{A^2}{\sum_{r=0}^{v} \alpha_r (z^{-r} + z^r)} = \frac{A^2/A_2}{1 + \frac{1}{A_2} \sum_{r=1}^{v} \alpha_r (z^{-r} + z^r)} \tag{12}$$

where $A_2 = \sum_{j=0}^{v} a_j^2$ and $\alpha_r = \sum_{j=r}^{v} a_j a_{j-r}$

Comparing (12) and (9) and assuming $\beta_1 = 1$ we have

$$\beta_j = \frac{\alpha_j}{\alpha_1} \tag{13}$$

$$k = 1 - \frac{A^2}{A_2} \tag{14}$$

$$B = -\frac{k A_2}{2\alpha_1} \tag{15}$$

$$\psi = \frac{1}{2\sigma^2} \frac{k}{1 - k} \tag{16}$$

These equations allow to obtain the prior parameters, ψ and β_j from the filter coefficients.

It is important to note that the inverse operation, i.e., the computation of a_j from the prior parameters ψ and β_j is much more difficult. For $v = 1$ and $v = 2$ it is possible to find closed form expressions. However, for $v > 2$ we need to solve a set of non linear equations, which is difficult to solve in a exact way.

3. IIR MAP SOLUTION

The MAP estimate is obtain by filtering the maximum likelihood estimate by a cascade of two filters: a causal and an anti causal filter. In practice, the ML signal is first filtered with the causal filter given by (11). The order of the output is then reversed and the signal is filtered again with the same filter as shown in Fig. 1

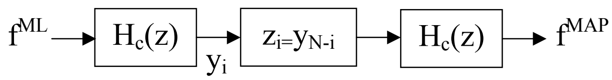

Fig. 1. MAP estimate. The signal is filtered twice using the same filter. A time-reversing operation is performed between both filtering operations.

4. EXPERIMENTAL RESULTS

In this section two examples of application are shown using synthetic and real data. The goal is to compare the reconstructions using the same observation model and priors of two different orders. The prior parameters ψ and β_j are computed from the filter coefficients, a_i, as well the parameter k. To make a fair comparison, the parameter k (and ψ) should have the same value in the first and higher order priors. The computation of ψ and k is performed by using (14) and (16).

4.1. 1D Synthetic Data

In this experiment, the data to be estimated is a 1D vector of 256 elements, with $f_i = 200, 64 \le i \le 200$ and $f_i = 64, 0 \le i \le 63 \wedge 201 \le i \le 256$. This vector was corrupted with additive Gaussian noise with distribution $N(0, 15^2)$ as shown in Fig. 2. To estimate F, the maximum likelihood estimates (in this case, the observations) are filtered twice (in both directions) with a 5-order linear filter, with poles at positions $p_0 = 0.4$ and $p_i = 0.4cos(\omega_i) \pm j0.5sin(\omega_i)$, with $\omega_1 = \pi/10+i(\pi/4-\pi/10)/2$ and $i = 0, 1, 2$. The coefficients, a_j, of the filters (11) are listed in Table 1 and the estimated vector is represented in Fig. 2 (thick line). Based on these coefficients, we have computed the prior parameters by using the equations (13)-(16). The β_j parameters are listed at Table 1 and the other parameters are $K = 0.996$ and $\psi = 0.3109$. Using $\psi = 0.3109$ a new estimation was performed using a first order Gibbs ([3]). These first order estimates are also shown in Fig. 2 (thin line). The frequency response of both filters are shown in Fig. 3.

Fig.2 shows a clear improvement of the transitions when the 5-order filter is used instead of the first order one. This improvement is confirmed with an increase of the signal to noise ratio from $SNR = 4dB$ to $SNR = 16.5dB$. The improvement at the transitions is explained by the frequency responses of the first order filter which has a larger bandwidth and shorter transition band (Fig.3).

4.2. MRI real data

In this experiment a MRI (Magnetic Resonance Imaging) image of 176×250 pixels is used. The 1D filter is applied to each column and the resulting image is then filtered again by lines.

index	a_i	β_i
0	1	-
1	-1.727	1
2	1.335	-0.515
3	-0.573	0.169
4	0.135	-0.033
5	-0.014	0.003

Table 1. Simulation results using synthetic data.

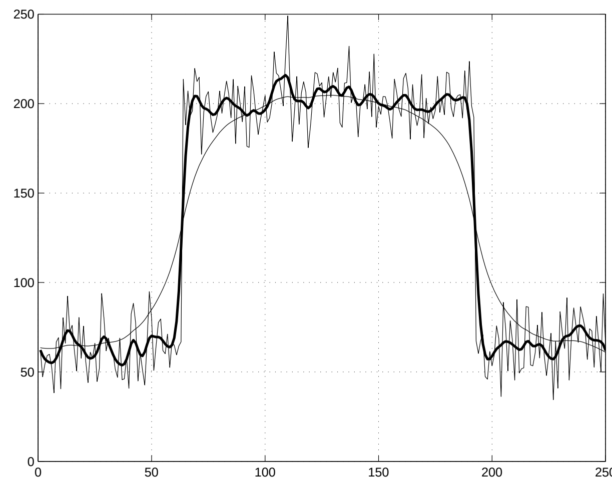

Fig. 2. Reconstruction results with synthetic data for $v = 1$ (thin line) and $v = 5$ (thick line) with $\psi = 0.3109$ and $K = 0.996$ in both experiments.

We still use a $5th$ order filter with poles at positions $p_0 = 0.25$ and $p_i = 0.25cos(\omega_i) \pm j0.5sin(\omega_i)$, with $\omega_1 = \pi/10 + i(\pi/4 - \pi/10)/2$ and $i = 0, 1, 2$. The coefficients, a_j, of the filters (11) and the wights β_j of the prior are listed at Table 2. The observed image corrupted by Gaussian noise and the filtered ones are shown in Fig. 4.

The parameters of the first order prior were computed from the coefficients a_j by using the equations (13)-(16), leading to $k = 0.9417$ and $\psi = 0.0128$. The pole position of the first order filter is $p_1 = 0.7047$.

The image obtained with the 5-order filter is sharper and shows a clear improvement of transitions as shown in Fig. 4. The signal to noise ratio, in this example, has also increased when the 5-order filter was used ($SNR = 19.3dB$) if compared with the first order filter ($SNR = 12.8dB$).

The frequency response of both filters are displayed in Fig: 5. As in the previous example the higher order filter preserves better the transitions due its larger bandwidth keeping the ability to reject the high frequency noise.

Furthermore, the proposed algorithms are very fast. They do not require an iterative optimization process but only two IIR filtering operations [7].

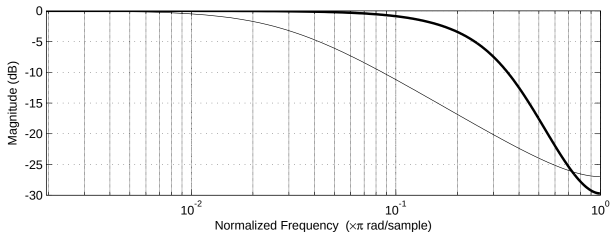

Fig. 4. MRI image of a knee. Left i)Noisy image. Middle ii)Restored image with the 6-order filter ($SNR = 19.3dB$). Right iii)Restored image with the correspondent first order prior ($SNR = 12.8dB$).

5. CONCLUSIONS

This paper proposes a fast algorithm to compute the MAP estimates of signals corrupted by noise. The proposed solution avoids the need of an iterative optimization procedure. The reconstructed signal is obtained by filtering the input signal with two IIR filters. The relationship between the filter and the Gibbs prior is defined. This allows to replace the design of the prior distribution by a filter design problem which can be addressed in the frequency domain.

Furthermore, it is shown that higher order priors lead to better performance at transitions than the popular first order prior used in many image reconstruction applications.

The design of the optimal filter for a given problem is

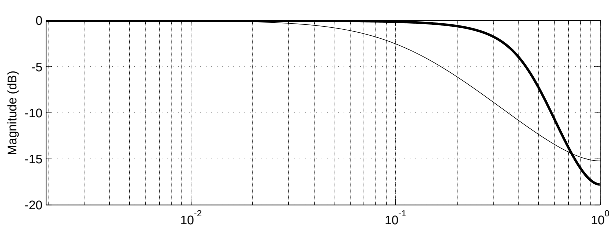

Fig. 5. Frequency response of the filter (used with real data) for $v = 1$(thin line) and $v = 5$ (thick line) with the same $\psi = 0.0128$ and $K = 0.9417$.

index	a_i	β_i
0	1	-
1	-1.079	1
2	0.678	-0.494
3	-0.250	0.156
4	0.052	-0.029
5	-0.005	0.002

Table 2. Simulation results using MRI data.

an open issue to be addressed in the future.

6. REFERENCES

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